ISSN:2636-7467(Online)

Volume 1
Issue 2
2018

international journal of MAPS IN MATHEMATICS

Editor in chief

Bayram SAHIN
Ege University, Izmir, TURKEY

International Journal of Maps in Mathematics is a fully refereed journal devoted to publishing recent results obtained in the research areas of maps in mathematics

www.journalmim.com

VOLUME 1 ISSUE 2
ISSN 2636-7467

September 2018
https://www.journalmim.com

International Journal of Maps in Mathematics

Editor-in-Chief

International Journal of Maps in Mathematics

Volume:1, Issue:2, (2018), Pages:137-153

ISSN: 2636-7467 (Online)

www.journalmim.com

CHARACTERIZATIONS OF CONTACT CR-WARPED PRODUCTS OF NEARLY COSYMPLECTIC MANIFOLDS IN TERMS OF ENDOMORPHISMS

WAN AINUN MIOR OTHMAN, SAYYEDAH A. QASEM, AND CENAP OZEL*

Abstract. The main objective of this paper is to characterize contact CR-warped product submanifolds of a nearly cosymplectic manifold in terms of endomorphisms T and F. We also obtain some neccessary and sufficient conditions for integarbility of distributions involve in the definition.

1. Introduction

For a submanifold M of an almost Hermitain $(\widetilde{M}, J, g)$, we decompose JU into tangential and normal components as $JU = TU + FU$, for any vector field U tangent to M. Many researchers including B.-Y. Chen described geometric properties of subamnifolds in terms of T and F [9]. Later, such characterizations were extended for warped products in almost Hermitian as well as almost contact settings in [1], [2], [3], [5], [9], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. In the present paper, we obtain some results on the characterization of contact CR-warped product submanifolds of a *nearly cosymplectic* manifold in terms of endomorphisms T and F.

Received:2018-02-26 *Revised:2018-05-15* *Accepted:2018-08-04*

2010 Mathematics Subject Classification:53C40,53C42, 53B25.

Contact CR-submanifold, warped products, nearly cosymplectic manifold, endomorphisms

* Corresponding author

The paper is organized as follows: In Section 2, we review some preliminary formulas and definitions. Section 3 is devoted to the study of contact CR-submanifold of a nearly cosymplectic manifold. In Section 4, we prove some lemmas on contact CR-warped product submanifolds of a *nearly cosymplectic* manifold, and then prove our main theorems on the characterization of warped product submanifolds in terms of the endomorphisms T and F.

2. Preliminaries

A $(2n+1)-$dimensional manifold $(\widetilde{M}, g)$ is said to be an *almost contact metric manifold* if it admits an endomorphism φ of its tangent bundle $T\widetilde{M}$, a vector field ξ, called *structure vector field* and η, the dual $1-$form of ξ satisfying the following.

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \varphi(\xi) = 0, \ \eta \circ \varphi = 0, \tag{2.1}$$

and

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \ \eta(U) = g(U, \xi), \tag{2.2}$$

for any U, V tangent to $\widetilde{M}$ [8]. An almost contact metric structure (φ, ξ, η) is said to be a normal if almost complex structure J on a product manifold $\widetilde{M} \times R$ given by

$$J(U, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(U)\frac{d}{dt}),$$

where f is a smooth function on $\widetilde{M} \times R$, has no torsion, i.e., J is integrable, the condition for normality in term of φ, η and ξ is $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ on $\widetilde{M}$, where $[\varphi, \varphi]$ is the *Nijenhius tensor* of φ. Finally, the second fundamental $2-$form Φ is defined by $\Phi(U, V) = g(U, \varphi V)$. An almost contact metric structure (φ, η, ξ) is said to be *cosymplectic* if it is normal and both Φ and η are closed. They *characterized* by $(\widetilde{\nabla}_U \varphi)Y = 0$ *and* $\widetilde{\nabla}_U \xi = 0$. An almost contact metric structure (φ, η, ξ) is said to be *nearly cosymplectic* if φ is killing, i.e., if

$$(\widetilde{\nabla}_U \varphi)U = 0 \text{ or equivalently } (\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = 0, \tag{2.3}$$

for any U, V tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ is the connection of the metric g on $\widetilde{M}$. If we replace $U = \xi$, $V = \xi$ in (2.3), we find that $(\widetilde{\nabla}_\xi \varphi)\xi = 0$ which is implies that $\varphi \widetilde{\nabla}_\xi \xi = 0$. Now applying φ and using (2.1), we get, $\widetilde{\nabla}_\xi \xi = 0$. Since from *Gauss formula* finally, we get $\nabla_\xi \xi = 0$ and $h(\xi, \xi) = 0$. The structure is said to be a *closely cosymplectic*, if φ is killing and η closed.

Now let M be a submanifold of $\widetilde{M}$. We will denote by ∇, the induced Riemannian connection on M and g, is the Riemannian metric on $\widetilde{M}$ as well as the metric induced on

M. Let TM and $T^{\perp}M$ be the Lie algebra of vector fields tangent to M and normal to M, respectively and $\nabla^{\perp}$ the induced connection on $T^{\perp}M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the *Gauss* and *Weingarten* formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.4}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.5}$$

for each $U,\ V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\widetilde{M}$. They are related as

$$g(h(U, V), N) = g(A_N U, V) \tag{2.6}$$

Now for any $U \in \Gamma(TM)$, we write

$$\varphi U = TU + FU, \tag{2.7}$$

where TU and FU are the tangential and normal components of φU, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$\varphi N = tN + fN, \tag{2.8}$$

where tN (resp. fN) is the tangential (resp. normal) component of φN. From (2.2) and (2.7), it is easy to observe that

$$g(TU, V) = -g(U, TV), \tag{2.9}$$

for each $U, V \in \Gamma(TM)$. The covariant derivatives of the endomorphism φ, T and F are defined, respectively as

$$(\widetilde{\nabla}_U \varphi)V = \widetilde{\nabla}_U \varphi V - \varphi \widetilde{\nabla}_U V, \quad \forall U, V \in \Gamma(T\widetilde{M}) \tag{2.10}$$

$$(\widetilde{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \quad \forall U, V \in \Gamma(TM) \tag{2.11}$$

$$(\widetilde{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \quad \forall U, V \in \Gamma(TM). \tag{2.12}$$

From [18] we have the following proposition

Proposition 2.1. *On any nearly cosymplectic manifold ξ is a killing form*

From the statement of above proposition we have the equality $g(\widetilde{\nabla}_U \xi, U) = 0$ for any vector field U tangent to nearly cosymplectic $\widetilde{M}$. We denote the tangential and normal parts of $(\widetilde{\nabla}_U \varphi)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ such that

$$(\widetilde{\nabla}_U \varphi)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{2.13}$$

for all U, V tangent to M. Making use of (2.2)-(2.12) in (2.13), we can easily obtain

$$\mathcal{P}_U V = (\widetilde{\nabla}_U T)V - A_{FV} U - th(U, V), \tag{2.14}$$

$$\mathcal{Q}_U V = (\widetilde{\nabla}_U F)V + h(U, TV) - fh(U, V). \tag{2.15}$$

Similarly for any $N \in \Gamma(T^\perp M)$, denoting the tangential and normal parts of $(\widetilde{\nabla}_U \varphi)N$ by $\mathcal{P}_U N$ and $\mathcal{Q}_U N$ such that

$$(\widetilde{\nabla}_U \varphi)N = \mathcal{P}_U N + \mathcal{Q}_U N. \tag{2.16}$$

Making use (2.3), (2.7), (2.8) in (2.16), we obtain

$$\mathcal{P}_U N = (\widetilde{\nabla}_U t)N + TA_N U - A_{fN} U \tag{2.17}$$

$$\mathcal{Q}_U N = (\widetilde{\nabla}_U f)N + h(U, tN) + FA_N U, \tag{2.18}$$

for all $U \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$. It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$,

$$\left.\begin{aligned} &(i)\ \mathcal{P}_{U+V} W = \mathcal{P}_U W + \mathcal{P}_V W,\ (ii)\ \mathcal{Q}_{U+V} W = \mathcal{Q}_U W + \mathcal{Q}_V W,\\ &(iii)\ \mathcal{P}_U(W + Z) = \mathcal{P}_U W + \mathcal{P}_U Z,\\ &(iv)\ \mathcal{Q}_U(W + Z) = \mathcal{Q}_U W + \mathcal{Q}_U Z,\\ &(v)\ g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W),\ (vi)\ g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N),\\ &(vii)\ \mathcal{P}_U \varphi V + \mathcal{Q}_U \varphi V = -\varphi(\mathcal{P}_U V + \mathcal{Q}_U V). \end{aligned}\right\} \tag{2.19}$$

In a nearly cosymplectic manifold $\widetilde{M}$, we have

$$(i)\ \mathcal{P}_U V + \mathcal{P}_V U = 0,\ (ii)\ \mathcal{Q}_U V + \mathcal{Q}_V U = 0, \tag{2.20}$$

for any $U, V \in \Gamma(T\widetilde{M})$.

3. Contact CR-submanifolds of a nearly cosymplectic manifold

Definition 3.1. *A submanifold M tangent to the structure vector filed ξ of an almost contact metric manifold $\widetilde{M}$ is said to be invariant if $\varphi(T_x M) \subseteq (T_x M)$ and anti-invariant if $\varphi(T_x M) \subseteq (T_x^\perp M)$ for each $x \in M$.*

Definition 3.2. *A submanifold M tangent to structure vector field ξ of an almost contact metric manifold $\widetilde{M}$ is said to be a contact CR-submanifold if there exist a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that*

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus < \xi >$, *where* $< \xi >$ *is 1-dimensional distribution spanned by* ξ,

(ii) *the distribution* $\mathcal{D}$ *is invariant, i.e.,* $\varphi(\mathcal{D}) \subseteq \mathcal{D}$,

(iii) *the distribution* $\mathcal{D}^{\perp}$ *is anti-invariant, i.e.,* $\varphi\mathcal{D}^{\perp}) \subseteq (T^{\perp}M)$.

If μ is an *invariant* subspace under φ of normal bundle $T^{\perp}M$. Then, in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = F\mathcal{D}^{\perp} \oplus \mu$. Let us denotes the orthogonal porojections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by B and C, respectively. Then for any $U \in \Gamma(TM)$, we have

$$U = BU + CU + \eta(U)\xi, \tag{3.21}$$

where $BU \in \Gamma(\mathcal{D})$ and $CU \in \Gamma(\mathcal{D}^{\perp})$. From (2.7), (2.8) and (3.21), we have

$$TU = \varphi BU, \; FU = \varphi CU. \tag{3.22}$$

So we observe the following equalities

$$\left.\begin{array}{l} (i)\; TC = 0, \;\; (ii)\; FB = 0, \\ (iii)\; t(T^{\perp}M) \subseteq \mathcal{D}^{\perp}\;, (iv)\; f(T^{\perp}M) \subseteq \mu. \end{array}\right\} \tag{3.23}$$

Theorem 3.1. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}\oplus < \xi >$ is integrable if and only if*

$$2g(\nabla_X Y, Z) = g(h(Y, \varphi X), \varphi Z) + g(h(X, \varphi Y), \varphi Z), \tag{3.24}$$

for any $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, then we derive

$$\begin{aligned} g([X, Y], Z) &= g(\widetilde{\nabla}_X Y, Z) - g(\widetilde{\nabla}_Y X, Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(\varphi\widetilde{\nabla}_Y X, \varphi Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(\widetilde{\nabla}_Y \varphi X - (\widetilde{\nabla}_Y \varphi)X, \varphi Z). \end{aligned}$$

From (2.4) and (2.3), we get

$$\begin{aligned} g([X,Y],Z) =& g(\widetilde{\nabla}_X Y, Z) - g(h(Y,\varphi X),\varphi Z) - g((\widetilde{\nabla}_X\varphi)Y,\varphi Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(h(Y,\varphi X),\varphi Z) - g(\widetilde{\nabla}_X\varphi Y,\varphi Z) \\ &+ g(\varphi\widetilde{\nabla}_X Y,\varphi Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(h(Y,\varphi X),\varphi Z) - g(h(X,\varphi Y),\varphi Z) \\ &+ g(\widetilde{\nabla}_X Y, Z) \\ &= 2g(\nabla_X Y, Z) - g(h(Y,\varphi X) + h(X,\varphi Y),\varphi Z). \end{aligned} \tag{3.25}$$

Our assertion follows from the above relation, which proves the theorem completely.

Lemma 3.1. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}\oplus <\xi>$ defines a totally geodesic foliation if and only if*

$$h(Y,\varphi X) + h(X,\varphi Y) \in \mu \tag{3.26}$$

for all $X,Y \in \Gamma(\mathcal{D}\oplus <\xi>)$.

Proof. The distribution $\mathcal{D}\oplus\xi$ is a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(\mathcal{D}\oplus\xi)$ for any $X,Y \in \Gamma(\mathcal{D}\oplus\xi)$. Applying these definition in the Eq 3.25, we get the required proof.

Similarly, for anti-invariant distribution, we have

Theorem 3.2. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^\perp$ is integrable if and only if*

$$2g(\nabla_Z W,\varphi X) = g(h(X,Z),\varphi W) + g(h(X,W),\varphi Z) \tag{3.27}$$

for all $Z,W \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(\mathcal{D}\oplus <\xi>)$.

Proof. Let us derive

$$\begin{aligned} g([Z,W],\varphi X) &= g(\widetilde{\nabla}_Z W,\varphi X) - g(\widetilde{\nabla}_W Z,\varphi X) \\ &= g(\widetilde{\nabla}_Z W,\varphi X) + g(\varphi\widetilde{\nabla}_W Z, X) \\ &= g(\widetilde{\nabla}_Z W,\varphi X) + g(\widetilde{\nabla}_W\varphi Z, X) - g((\widetilde{\nabla}_W\varphi)Z, X), \end{aligned}$$

For any $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D} \oplus <\xi>)$. From (2.4), (2.5) and (2.3), we obtain

$$\begin{aligned} g([Z,W],\varphi X) &= g(\nabla_Z W, \varphi X) - g(A_{\varphi Z}W, X) + g((\widetilde{\nabla}_Z \varphi)W, X) \\ &= g(\nabla_Z W, \varphi X) - g(A_{\varphi Z}W, X) + g(\widetilde{\nabla}_Z \varphi W, X) - g(\varphi \widetilde{\nabla}_Z W, X) \\ &= g(\nabla_Z W, \varphi X) - g(A_{\varphi Z}W, X) - g(A_{\varphi W}Z, X) + g(\nabla_Z W, \varphi X) \\ &= 2g(\nabla_Z W, \varphi X) - g(A_{\varphi Z}W, + A_{\varphi W}Z, X). \end{aligned} \tag{3.28}$$

Thus the desired result follows from the last the relation. It completes the proof of the theorem.

The following corollary is a consequence of the Theorem 3.2,

Corollary 3.1. *The anti-invariant distribution $\mathcal{D}^{\perp}$ of contact CR-submanifold M in a nearly cosymplectic manifold $\widetilde{M}$ is defines totally geodesic foliation if and only if*

$$A_{\varphi Z}W + A_{\varphi W}Z \in \Gamma(\mathcal{D}^{\perp}) \tag{3.29}$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$.

Proof. The proof follows from (3.28) and the definition of totally geodesic foliation.

Theorem 3.3. *The distribution $\mathcal{D}^{\perp}$ of a contact CR-submanifold M in a nearly cosymplectic manifold $\widetilde{M}$ is integrable if and only if*

$$g(\mathbf{P}_Z W, \varphi X) = 2\eta(X) g(\widehat{\nabla}_Z \xi, W)$$

or equivalent

$$g(A_{\varphi Z}W, \varphi X) = g(A_{\varphi W}Z, \varphi X), \tag{3.30}$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D} \oplus <\xi>)$.

Proof. Let use the definition of Lie bracket, then simplification gives

$$g([Z,W], X) = g(\widetilde{\nabla}_Z W - \widetilde{\nabla}_W Z, X),$$

for $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D} \oplus <\xi>)$. Using (2.2), we get

$$g([Z,W], X) = g(\varphi \widetilde{\nabla}_Z W - \varphi \widetilde{\nabla}_W Z, \varphi X) - \eta(X) g(\widehat{\nabla}_Z \xi, W) + \eta(X) g(\widehat{\nabla}_W \xi, Z).$$

Hence, using the property of covariant derivative (2.10), structure equation of a nearly cosymplectic manifold (2.3) and Proposition 2.1, we obtain

$$g([Z,W],X) = g(2\mathcal{P}_Z W - \widetilde{\nabla}_W \varphi Z + \widetilde{\nabla}_Z \varphi W, \varphi X) - 2\eta(X) g(\widehat{\nabla}_Z \xi, W).$$

Now from Weingarten formula (2.5), we have

$$g([Z,W],X) = g(2\mathcal{P}_Z W, \varphi X) - g(A_{\varphi Z} W - A_{\varphi W} Z, \varphi X) - 2\eta(X) g(\widehat{\nabla}_Z \xi, W),$$

which proves the our assertion. It compete proof of the Theorem.

4. Contact CR-warped products of nearly cosymplectic manifolds

The warped product manifolds are the generalized version of Riemannian product manifolds. The notion of warped product manifold defined as follows:

Let (B, g_1) and (F, g_2) be two Riemannian manifolds and f, a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $B \times F = (B \times F, g)$, where $g = g_1 + f^2 g_2$. A warped product manifold M is said to be a trivial warped product if its warping function f is constant. A trivial warped product $B \times F$ is nothing but Riemannian product $B \times_f F$ where ${}_fF$ is the Riemannian manifold with Riemannian metric $f^2 g_F$ which is homothetic to the original metric g_F of F. Bishop and O'Neill [7] also obtained the following lemma which provides some basic formulas on warped product manifolds

Lemma 4.1. *Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$ then*

(i) $\nabla_X Y \in \Gamma(TB)$,

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f) Z$,

(iii) $\nabla_Z W = \nabla'_Z W - g(Z,W) \nabla \ln f$,

where $\nabla \ln f$ is gradient of the function $\ln f$ which is defined as $g(\nabla \ln f, X) = X \ln f$, for any $X \in \Gamma(TB)$. Moreover, ∇ and ∇' are the Levi-Civitas connection on B and F, respectively.

It follows from Lemma 4.1 that B is totally geodesic submanifold in M and F is totally umbilical submanifold in M. In this way, we investigate the characterization of non-trivial warped product submanifolds $M_T \times_f M_\perp$ of nearly cosymplectic manifolds in terms of T and F. In terms tensor fields we have following characterization results.

Theorem 4.1. [9] *A CR-submanifold M of a Kaehler manifold $\widetilde{M}$ is a CR-product if and only if T is parallel, i.e.,*

$$\widetilde{\nabla}T = 0.$$

Theorem 4.2. [12] *A proper contact CR-submanifold M of a Kaehler manifold $\widetilde{M}$ is locally CR-warped product if and only if T satisfies:*

$$(\widetilde{\nabla}_U T)V = (TBU\mu)CU + g(CU, CV)J\nabla\mu$$

any $U, V \in \Gamma(TM)$, where C and B are the projections on $\mathcal{D}^{\perp}$ and $\mathcal{D}$, respectively.

In the proceeding these study, we derive the following results which are very important for proving the characterization theorem.

Lemma 4.2. *Let $M = M_T \times_f M_{\perp}$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then*

(i) $(\widetilde{\nabla}_X T)Z = 0$, *(ii)* $(\widetilde{\nabla}_Z T)X = (TX\ln f)Z$,

(iii) $(\widetilde{\nabla}_\xi T)X = T\nabla_X\xi$, *(iv)* $(\widetilde{\nabla}_U T)\xi = -T\nabla_{BU}\xi$,

(v) $(\widetilde{\nabla}_U T)Z = g(CU, Z)T\nabla\ln f$.

for all $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_{\perp})$ and $U \in \Gamma(TM)$.

Proof. First part directly follows from (2.11), Lemma 4.1(ii) and using the fact that $TZ = 0$, $\forall\ Z \in \Gamma(TM_{\perp})$. For the second part, we find

$$\begin{aligned}(\widetilde{\nabla}_Z T)X &= \nabla_Z TX - T\nabla_Z X\\ &= (TX\ln f)Z - (X\ln f)TZ\\ &= (TX\ln f)Z,\end{aligned}$$

which is (ii). Similarly, to prove (iii), we have

$$(\widetilde{\nabla}_U T)Z = \nabla_U TZ - T\nabla_U Z. \tag{4.31}$$

Since $TZ = 0$, $\forall\ Z \in \Gamma(TM_{\perp})$ and using (3.21) in (4.31), we obtain

$$(\widetilde{\nabla}_U T)Z = -T\{\nabla_{BU}Z + \nabla_{CU}Z + \eta(U)\nabla_\xi Z\}.$$

From Lemma 4.1(ii), we derive

$$(\widetilde{\nabla}_U T)Z = -(BU\ln f)TZ - T\nabla_{CU}Z - \eta(U)(\xi\ln f)TZ.$$

Since $\xi \ln f = 0$, funded by [18], then using Lemma 4.1(iii), it is easily obtain that

$$(\widetilde{\nabla}_U T)Z = -T\{\nabla'_{CU} Z - g(CU, Z)\nabla \ln f\}$$
$$= g(CU, Z)T\nabla \ln f.$$

Now, for any $X, Y \in \Gamma(TM_T)$, then from (2.14) and (2.20)(i), we get

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 2th(X, Y). \tag{4.32}$$

By equation (2.11) and the fact that M_T is totally geodesic in M, it follows that $(\widetilde{\nabla}_X T)Y$ lies in M_T, thus left hand side in (4.32) completely lies in M_T. Therefore equating the tangential components along M_T in las equation, we get $th(X, Y) = 0$, which means that $h(X, Y) \in \Gamma(\mu)$. Then from (4.32), we find

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 0. \tag{4.33}$$

If, we set $Y = \xi$ in (4.33), we simplifies

$$(\widetilde{\nabla}_X T)\xi + (\widetilde{\nabla}_\xi T)X = 0$$
$$(\widetilde{\nabla}_\xi T)X = T\nabla_X \xi,$$

which gives the third result of the lemma. It completes proof of lemma.
First characterization theorem in terms of ∇T.

Theorem 4.3. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$ with both invariant and anti-invariant distributions are integrable. Then M is locally a CR-warped product if and only if*

$$(\widetilde{\nabla}_U T)U = (TBU\lambda)CU + ||CU||^2 T\nabla\lambda, \tag{4.34}$$

or equivalently

$$(\widetilde{\nabla}_U T)V + (\widetilde{\nabla}_V T)U = (TBV\lambda)CU + (TBU\lambda)CV + 2g(CU, CV)T\nabla\lambda, \tag{4.35}$$

for each $U, V \in \Gamma(TM)$ and λ is a C^∞-function on M satisfying $Z\lambda = 0$, for each $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Assume that M be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then applying (3.21) in $(\widetilde{\nabla}_U T)U$, we derive

$$(\widetilde{\nabla}_U T)U = (\widetilde{\nabla}_U T)BU + (\widetilde{\nabla}_U T)CU + \eta(U)(\widetilde{\nabla}_U T)\xi.$$

Again applying (3.21) and using Lemma 4.2(iv), we get

$$(\widetilde{\nabla}_U T)U = (\widetilde{\nabla}_{BU} T)BU + (\widetilde{\nabla}_{CU} T)BU + (\widetilde{\nabla}_U T)CU + \eta(U)(\widetilde{\nabla}_\xi T)TU - \eta(U)T\nabla_{BU}\xi.$$

As M_T is totally geodesic in M, then the first term of right side in the above equation is zero by using (2.3) and from the Lemma 4.2(ii), (iii), (v), we arrive at

$$(\widetilde{\nabla}_U T)U = (TBU\lambda)CU + ||CU||^2 T\nabla\lambda,$$

where $\lambda = \ln f$. Hence, we obtain desire result (4.34). Furthermore, the equation (4.35) can be easily derive by replacing U by $U+V$ in (4.34).

Conversely, suppose that M is a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$ such that condition (4.35) holds. Then choosing $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and using the fact that $CX = CY = 0$ in (4.35), we get the following condition, i.e.,

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 0. \tag{4.36}$$

Thus, from (2.20)(i), for nearly cosymplectic $\widetilde{M}$,

$$\mathcal{P}_X Y + \mathcal{P}_Y X = 0; \tag{4.37}$$

From (4.36), (4.37) and (2.17), we can easily obtain the condition $th(X, Y) = 0$, which is implies that $h(X, Y) \in \mu$ for all $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$. Then using the integrability of $\mathcal{D}\oplus < \xi >$ and Theorem 3.1, which indicate that $g(\nabla_X Y, Z) = 0$, for all $Z \in \Gamma(\mathcal{D}^\perp)$. This proves that $\mathcal{D}\oplus < \xi >$ is parallel and each of its leaves M_T is totally geodesic in M. Furthermore, using the fact $BZ = BW = 0$, we get

$$(\widetilde{\nabla}_Z T)W + (\widetilde{\nabla}_W T)Z = 2g(Z, W)T\nabla\lambda, \tag{4.38}$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp)$. From (2.11), we have

$$(\widetilde{\nabla}_Z T)W + (\widetilde{\nabla}_W T)Z = A_{FZ}W + A_{FW}Z + 2th(Z, W). \tag{4.39}$$

Thus by (4.38) and (4.39), it follows that

$$A_{FZ}W + A_{FW}Z + 2th(Z, W) = 2g(Z, W)P\nabla\lambda. \tag{4.40}$$

Taking the inner product in (4.40) with $X \in \Gamma(D\oplus < \xi >)$, we obtain

$$g(A_{FZ}W, X) + g(A_{FW}Z, X) + 2g(th(Z, W), X) = 2g(Z, W)g(T\nabla\lambda, X). \tag{4.41}$$

The second term of right hand side in (4.41) is zero from (3.23)(iii), that is

$$g(h(X,W),\varphi Z)+g(h(X,Z),\varphi W)=2g(Z,W)g(T\nabla\lambda,X). \tag{4.42}$$

From the hypothesis of theorem that we assumed the totally real distribution is integrable. Then necessary and sufficient condition for integrability of $\mathcal{D}^{\perp}$ from the Theorem 3.2 and using (4.42), it follows that

$$\begin{aligned} g(\nabla_Z W,\varphi X) &= g(Z,W)g(T\nabla\lambda,X) \\ &= -g(Z,W)g(\nabla\lambda,\varphi X). \end{aligned} \tag{4.43}$$

As $\mathcal{D}^{\perp}$ is assumed to be integrable, then the second fundamental form of the immersion of $M_{\perp}(leafof\mathcal{D}^{\perp})$ into M is denoted by $h^{\perp}$. Hence, in point view Gauss formula (2.4) in (4.43), i.e.,

$$g(h^{\perp}(Z,W),\varphi X)=-g(Z,W)g(\nabla\lambda,\varphi X),$$

which is implies that

$$h^{\perp}(Z,W)=-g(Z,W)\nabla\lambda.$$

It means that $M_{\perp}$ is totally umbilical in M with mean curvature vector $H^{\perp}=-\nabla\lambda$. Now we can easily prove that $H^{\perp}$ is parallel corresponding to the normal connection ∇' of $M_{\perp}$ in M, i.e., $Z(\lambda)=0$ for all $Z\in\Gamma(D^{\perp})$ and $\nabla_Y\nabla\lambda\in\Gamma(\mathcal{D}\oplus<\xi>)$. Hence, the leaves of $\mathcal{D}^{\perp}$ are extrinsic spheres in M. From result of [11], we conclude that M is a warped product submanifold. The proof is done.

Lemma 4.3. *Let $M=M_T\times_f M_{\perp}$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then*

(i) $g((\widetilde{\nabla}_X F)Y,\varphi W)=0$, *(ii)* $g((\widetilde{\nabla}_X F)Z,\varphi W)=0$,

(iii) $g((\widetilde{\nabla}_Z F)X,\varphi W)=-(X\ln f)g(Z,W)$, *(iv)* $g((\widetilde{\nabla}_\xi F)Z,\varphi W)=0$,

(v) $g((\widetilde{\nabla}_Z F)W',\varphi W)=g(\mathbf{Q}_Z W',\varphi W)$,

for any $X,Y\in\Gamma(TM_T)$ and $Z,W,W'\in\Gamma(TM_{\perp})$.

Proof. Let M be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$, then,

$$\begin{aligned} g((\widetilde{\nabla}_X F)Y,\varphi W) &= g(-F\nabla_X Y,\varphi W) \\ &= -g(\nabla_X Y,W). \end{aligned}$$

As M_T is totally geodesic in M, then from the above equation, we get (i). To the other parts, from (2.18), it is easily seen that

$$((\widetilde{\nabla}_X F)Z, \varphi W) = g((\mathcal{Q}_X Z + fh(X,Z), \varphi W). \tag{4.44}$$

Using nearly cosymplectic manifold (2.3), and the property (v),(vii) of (2.19) in equation (4.44), we obtain

$$((\widetilde{\nabla}_X F)Z, \varphi W) = g(\varphi X, \mathcal{P}_Z W).$$

Then integrability Theorem 3.3 of the distribution $\mathcal{D}^{\perp}$, gives

$$((\widetilde{\nabla}_X F)Z, \varphi W) = 2\eta(X) g(\widehat{\nabla}_Z \xi, W) = 2\eta(X)(\xi \ln f) g(Z, W),$$

which is the result (ii) of lemma. Again, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$, we obtain

$$((\widetilde{\nabla}_Z F)X, \varphi W) = -g(F\nabla_Z X, \varphi W).$$

From Lemma 4.1(ii), we obtain (iii) as follows

$$g((\widetilde{\nabla}_Z F)X, \varphi W) = -(X \ln f) g(Z, W).$$

Now to prove (v), from (2.18), we find that

$$g((\widetilde{\nabla}_Z F)W', \varphi W) = g(\mathcal{Q}_Z W', \varphi W).$$

Similarly, we obtain

$$\begin{aligned} g((\widetilde{\nabla}_\xi F)Z, \varphi W) &= g(\mathcal{Q}_\xi Z + fh(\xi, Z), \varphi W) \\ &= g(\mathcal{Q}_\xi Z, \varphi W). \end{aligned}$$

Using the property (2.19)(vi), we can derive

$$g((\widetilde{\nabla}_\xi F)Z, \varphi W) = g(\varphi \xi, \mathcal{P}_Z W)$$
$$g((\widetilde{\nabla}_\xi F)Z, \varphi W) = 0,$$

which is the last result. It completes proof of the lemma.

Similarly, the second characterization theorem in terms of ∇F.

Theorem 4.4. *Assume that M be a contact CR-submanifold in a nearly cosymplectic manifold $\widetilde{M}$ with anti-invariant and invariant distributions are integrable. Then the M is locally a CR-warped product if only if*

$$g((\widetilde{\nabla}_U F)U, \varphi W) = -(BU\lambda) g(CU, W) \tag{4.45}$$

or equivalently

$$g((\widetilde{\nabla}_U F)V + (\widetilde{\nabla}_V F)U, \varphi W) = -(BU\lambda)g(CV, W) - (BV\lambda)g(CU, W) \tag{4.46}$$

for each $U, V \in \Gamma(TM)$ *and* λ *is a* C^∞*-function on* M *satisfying* $Z\lambda = 0$ *for each* $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let M be a contact CR-warped product submanifold in a nearly cosymplectic manifold $\widetilde{M}$. The property (3.21) gives

$$\begin{aligned} g((\widetilde{\nabla}_U F)V, \varphi W) =& g((\widetilde{\nabla}_{BU} F)BV, \varphi W) + g((\widetilde{\nabla}_{CU} F)BV, \varphi W) \\ &+ \eta(U)g((\widetilde{\nabla}_\xi F)BV, \varphi W) + g((\widetilde{\nabla}_{BU} F)CV, \varphi W) \\ &+ g((\widetilde{\nabla}_{CU} F)CV, \varphi W) + \eta(U)g((\widetilde{\nabla}_\xi F)CV, \varphi W) \\ &+ \eta(V)g((\widetilde{\nabla}_U F)\xi, \varphi W). \end{aligned}$$

Using Lemma 4.3, we obtain

$$g((\widetilde{\nabla}_U F)V, \varphi W) = g(\mathcal{Q}_{CU} CV, \varphi W) - (BV\lambda)g(CU, W). \tag{4.47}$$

By the polarization identity, we get

$$g((\widetilde{\nabla}_V F)U, \varphi W) = g(\mathcal{Q}_{CV} CU, \varphi W) - (BU\mu)g(CV, W). \tag{4.48}$$

From (4.47), (4.48) and (2.20)(ii), we get required result (4.46) or in particular, if we replace $V = U$ in (4.46) and using the property of nearly cosymplectic structure, i.e., $\mathcal{Q}_U U = 0$, we get first desired result of the theorem.

Conversely, let us consider that M be a CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$ with the condition (4.46) holds. Then using the fact that $CX = CY = 0$, in (4.46), simplification gives

$$g((\widetilde{\nabla}_X F)Y + (\widetilde{\nabla}_Y F)X, \varphi W) = 0,$$

for each $X, Y \in \Gamma(\mathcal{D}\oplus <\xi>)$. Thus, from the relations (2.18) and (2.20)(ii), we derive

$$2g(fh(X, Y), \varphi W) - g(h(X, TY) + h(Y, TX), \varphi W) = 0.$$

From the hypothesis of theorem, i.e., the distribution $(\mathcal{D}\oplus <\xi>)$ is integrable, then from the Theorem 3.1 gives $g(\nabla_X Y, W) = 0$, for all $W \in \Gamma(\mathcal{D}^\perp)$ which is implies that $\nabla_X Y \in (\mathcal{D}\oplus <\xi>)$. It means that the invariant distribution $(\mathcal{D}\oplus <\xi>)$ is a totally geodesic in M, i.e., the leaves of $(\mathcal{D}\oplus <\xi>)$ in M are totally geodesic. Similarly, other part, we have

$$g((\widetilde{\nabla}_X F)Z + (\widetilde{\nabla}_Z F)X, \varphi W) = -(X\lambda)g(Z, W),$$

for any $Z \in \Gamma(\mathcal{D}^{\perp})$, $X \in \Gamma(\mathcal{D} \oplus < \xi >$ and from (4.46). Then relations (2.12) and (2.18), we derive

$$g(\mathcal{Q}_X Z, \varphi W) - g(F\nabla_Z X, \varphi W) = -(X\lambda)g(Z, W).$$

On the other hand, the anti-invariant distribution $\mathcal{D}^{\perp}$ is integrable by hypothesis of the theorem. Thus first term of left hand side identicaly zero by the Theorem 3.3, then the above equation takes the form

$$g(\nabla_Z W, X) = -g(Z, W)g(\nabla\lambda, X).$$

Let $M_{\perp}$ denote the leaves of $\mathcal{D}^{\perp}$. If h' denotes the second fundamental form of the immersion of $M_{\perp}$ into M, then by the Gauss formula (2.4), we can write as

$$g(h'(Z, W), X) = -g(Z, W)g(\nabla\lambda, X),$$

which means that

$$h'(Z, W) = -g(Z, W)\nabla\lambda.$$

It implies that $M_{\perp}$ is totally umbilical in M with mean curvature vector $H = -\nabla\lambda$. Now we shall prove that H is parallel corresponding to the normal connection $\mathcal{D}$ of $M_{\perp}$ in M. In similar way of the Theorem 4.3, this means that the leaves of $\mathcal{D}^{\perp}$ are extrinsic spheres in M. Then by result of [11], M is locally a warped product. It completes proof the theorem.

Acknowledgements

The author is thankful to Dr. Akram Ali for the discussion as regards. Also they are thankful to the reviewers for their valuable suggestions, which really improved the quality of the manuscript. This work is supported to University of Malaya research grant FP074-2015A.

References

[1] N. S. Al-Luhaibi, Falleh R. Al-Solamy and V. A. Khan, *CR-warped product submanifolds of nearly Kaehler manifolds*, J. Korean. Math. Soc. 46 (2009)no-5, pp. 979-995.

[2] A. Ali, P. Laurian-Ioan, *Geometry of warped product immersion of Kenmotsu space forms and its applications to slant immersions.* J. Geom. Phys 114 (2017) 276-290.

[3] A. Ali and C. Ozel, *Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications.* Int. J. Geom. Methods Mod. Phys. 14, 1750042 (2017).

[4] A. Ali., W. A. M. Othman, and C. Ozel., *Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds.* J. Inequal. Appl. 2015, Article ID 9 (2015)

[5] A. Ali; W. A. M. Othman and C. Ozel, *Characterizations of contact CR-warped product submanifolds of nearly Sasakian manifolds*, Balkan J. Geom and Appl., vol.21, No.2, (2016), pp. 9-20,

[6] A. Ali and W. A. M. Othman, *Geometric aspects of CR-warped product submanifolds of $T-$manifolds,* Asian-European J. Math 0, 1750067 (2017).

[7] R. L .Bihop and B. O'Neil, *Manifold of negative curvature*, Trans. Amer. Math. Soc. 145 (1969) 1-49.

[8] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, New York, 1976.

[9] B. Y. Chen, *CR-subamnifold of a Kaehler manifold*, I, J. Differential Geom. 16(1981), no-2, 305-322.

[10] B. Y. Chen, *Geometry of warped product CR-submanifold in Kaehler manifolds*, Monatsh. Math. 133 (2001), no.03, 177-195.

[11] S. Hiepko, *Eine inner kennzeichungder verzerrten produckt*, Math, Ann. 244 (1979), 209-215

[12] V. A. Kham, K. A. Khan and S. Uddin, *CR-warped product submanifolds in a Kaehler manifolds*, South-east. Asian. Bull. Math. Vol 33, No 5 (2009), 865-874.

[13] V. A. Khan and M. Shuib, *Some warped product submanifolds of a kenmutso manifold*, Bull. Korean Math. Soc. Vol 51No-3, (2014) 863-881.

[14] M. I. Munteanu,*Warped product contact CR-submanifold of Sasakian space forms*, Publ. Math. Debrecen, 66(6) (2005), 75-120.

[15] B. Sahin and M. Atceken, *Semi-invariant submanifolds of Riemannian product manifolds*, Balkan J. Geom. Appl. 8 (2003) 91-100.

[16] B. Sahin, *Slant submanifolds of an almost product Riemannian manifold*, J. Korean Math. Soc. 43 (2006) 717-732.

[17] B. Sahin, *Warped product semi-invariant submanifolds of locally product Riemannian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie, Tome 49 (97) (2006) 383-394.

[18] S. Uddin, S. H. Kon, M. A. Khan, K. Singh, *Warped product semi-invariant submanifolds of nearly cosymplectic manifold*, Math. Prob. Enng. Article ID-230374(2011).

[19] S. Uddin and A.Y.M. Chi, *Warped product pseudo-slant submanifolds of nearly Kaehler manifolds,* An. St. Univ. Ovidius Constanta. 19 (3) (2011), 195-204.

[20] S. Uddin, F.R. Al-Solamy and K.A. Khan, *Geometry of warped product pseudo-slant submanifolds in nearly Kaehler manifolds,* An. Stiint.Univ. Al. I. Cuza Iasi Sect. I a Mat., Tom LXII, 3 (2016), 223-234.

[21] S. Uddin and F.R. Al-Solamy, *Warped product pseudo-slant submanifolds of cosymplectic manifolds*, An. Stiint. Univ. Al. I. Cuza Iasi Mat (N.S.) Tome LXIII (2016), f2 vol. 3, 901-913.

[22] S. Uddin and F.R. Al-Solamy, *Warped product pseudo-slant immersions in Sasakian manifolds,* Publ. Math. Debrecen, 91 (3-4) (2017), 331-348.

[23] S. Uddin, B.-Y. Chen and F.R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds,* Mediterr. J. Math. (2017) 14: 95. doi:10.1007/s00009-017-0896-8.

[24] S. Uddin, *Geometry of warped product semi-slant submanifolds of Kenmotsu manifolds*, Bull. Math. Sci. (2017), doi: 10.1007/s13373-017-0106-9

Institute of Mathematical Science, Faculty of Sciences, University of Malaya
Email address: wanainun@um.edu.my

Institute of Mathematical Science, Faculty of Sciences, University of Malaya
Email address: $sayydah_sharabi@yahoo.com$

Department of Mathematics, King Abdul Aziz University Jeddah Saudi Arabia.
Email address: cenap.ozel@gmail.com

International Journal of Maps in Mathematics

Volume:1, Issue:2, (2018), Pages:154-161

ISSN: 2636-7467 (Online)

www.journalmim.com

SOME REMARKS ON THE GENERALIZED MYERS THEOREMS

YASEMIN SOYLU

ABSTRACT. In this paper, firstly, we prove a generalization of Ambrose (or Myers) theorem for the Bakry-Emery Ricci tensor. Later, we improve the diameter estimate obtained by Galloway for complete Riemannian manifolds. To obtain these results, we utilize the Riccati inequality and the index form of a minimizing unit speed geodesic segment, respectively.

1. INTRODUCTION

Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ and let f be a smooth function on M. By the Bakry-Emery Ricci tensor we mean

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess} f, \tag{1.1}$$

where Ric and Hess$f$ are the Ricci tensor and the Hessian of f, respectively [2].

When f is a constant function, the Bakry-Emery Ricci tensor becomes the original Ricci tensor. We recall Ambrose's result [1], which gives an important generalization of the Myers compactness theorem [13] for the original Ricci tensor as another variant.

Theorem 1.1. *[1] If there exists a point $p \in M$ such that the condition*

$$\int_0^\infty \mathrm{Ric}(\gamma'(t), \gamma'(t))dt = \infty \tag{1.2}$$

Received:2018-02-27 *Accepted:2018-05-09*

2010 Mathematics Subject Classification.53C20, 53C21.

Key words: Bakry-Emery Ricci tensor, Index form, Myers-type theorem, Riccati inequality

holds along every geodesic $\gamma(t)$ emanating from $p \in M$, then manifold is compact.

In [19], Zhang proved the Ambrose's compactness theorem for the Bakry-Emery Ricci tensor given in (1.1).

Theorem 1.2. *[19] If there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from p satisfies*

$$\int_0^\infty \mathrm{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \tag{1.3}$$

and $f(x) \leq C(d(x,p)+1)$ for some constant C, where $d(x,p)$ is the distance from p to x, then M is compact.

Another generalization has been considered by Cavalcante-Oliveira-Santos in [3], where the condition on f given in Theorem 1.2 is replaced with a condition on the derivation of f as follows:

Theorem 1.3. *[3] Suppose that there exists a point p in a complete manifold M such that every geodesic $\gamma(t)$ emanating from p satisfies*

$$\int_0^\infty \mathrm{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \tag{1.4}$$

and $\frac{df}{dt} \leq 0$. Then M is compact.

The proofs of the above theorems are based on the Riccati inequality and a careful analysis of this inequality being different from calculus of variations. Moreover, these theorems do not require that the original Ricci tensor and the Bakry-Emery Ricci tensor be everywhere non-negative. However, these results cannot give an upper bound for the diameter of a manifold.

Our first aim is to improve condition on the function f under the same Ric_f assumption as in the Theorem 1.3.

On the other hand, Galloway [6] proved a perturbed version of Myers compactness theorem by the derivative in the radial direction of some bounded function as follows:

Theorem 1.4 (Galloway)**.** *Let M be a complete Riemannian manifold and γ be a geodesic joining two points of M. Suppose that*

$$\mathrm{Ric}(\gamma'(t), \gamma'(t)) \geq a + \frac{d\phi}{dt} \tag{1.5}$$

holds along γ for some constant $a > 0$, and $|\ \phi\ | \leq c$ for some constant $c \geq 0$. Then M is compact and

$$\operatorname{diam}(M) \leq \frac{\pi}{a}\big(c + \sqrt{c^2 + a(n-1)}\big). \tag{1.6}$$

Our second aim is to show that there is a sharper diameter estimate than Galloway's diameter estimate (1.6).

We are now ready to give our main theorems.

Theorem 1.5. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$. Suppose there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from p satisfies*

$$\int_0^\infty \operatorname{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \tag{1.7}$$

and $f'(t) \leq \frac{1}{4}(1 - \frac{1}{t})$ for all $t \geq 1$, then manifold is compact.

In the above theorem, we provide that the condition $f'(t) \leq 0$ given in Theorem 1.3 for $t = 1$. In order to prove Theorem 1.5, we use the Riccati inequality.

Theorem 1.6. *Let (M, g) be a complete Riemannian manifold and γ be a geodesic joining two points of M. Suppose that*

$$\operatorname{Ric}(\gamma'(t), \gamma'(t)) \geq a + \frac{d\phi}{dt} \tag{1.8}$$

holds along γ for some constant $a > 0$, and $|\ \phi\ | \leq c$ for some constant $c \geq 0$. Then M is compact and

$$\operatorname{diam}(M) \leq \frac{1}{a}\Big(2c + \sqrt{4c^2 + a(n-1)\pi^2}\Big). \tag{1.9}$$

The diameter estimate (1.9) above is sharper than (1.6) by Galloway. In order to prove above theorem, we use the index form of a minimizing unit speed geodesic segment. For basic facts about this topic, we refer the reader to the book [8, 14]).

Remark 1.1. *There exists many varied examples of compactness theorems involving the original Ricci tensor and modified Ricci tensors; see for instance [4, 5, 7, 9–12, 15–18].*

2. Proofs of the Theorems

Before stating our main results, we recall the definitions of gradient, Hessian and Laplacian of any smooth function $f \in \mathcal{C}^\infty(M)$ on a Riemannian manifold. The gradient, Hessian and Laplacian are defined by

$$g(\nabla f, V) = V(f), \quad \big(\operatorname{Hess}(f)\big)(V, W) = g(\nabla_V \nabla f, W) \ \text{ and } \ \Delta f = \operatorname{tr}\big(\nabla\nabla f\big) \tag{2.10}$$

for all vector fields V, W, respectively. The Riemannian curvature tensor is defined as

$$R(V,W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z, \tag{2.11}$$

and the Ricci curvature as

$$\mathrm{Ric}(V,W) = \sum_{i=1}^{n} g(R(E_i,V)W, E_i) \tag{2.12}$$

for all vector fields V, W, Z, where $\{E_i\}_{i=1}^n$ is an orthonormal frame of (M,g) Riemannian manifold.

Proof of Theorem 1.5. We assume that M is a non-compact Riemannian manifold and let $\gamma(t)$ be an unit speed ray starting from p. For every $t > 0$, $m(t)$ denotes the Laplacian of distance function from a fixed point $p \in M$. We know from some calculations with the Bochner formula that this gives the following Riccati inequality

$$m'(t) + \frac{1}{n-1} m^2(t) + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0. \tag{2.13}$$

We consider a smooth function $F(t)$ defined by

$$F(t) := m(t) + \zeta(t) \tag{2.14}$$

for all $t > 0$, where $\zeta \in \mathcal{C}^\infty(M)$. The derivation of $F(t)$ gets

$$F'(t) = m'(t) + \zeta'(t). \tag{2.15}$$

Combining (2.13) and (2.15), we obtain

$$F'(t) - \zeta'(t) + \frac{1}{n-1} m^2(t) + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0. \tag{2.16}$$

It is clear that we have

$$m(t) = F(t) - \zeta(t), \tag{2.17}$$

by (2.14). Substituting (2.17) into (2.16), we obtain

$$F'(t) - \zeta'(t) + \frac{1}{n-1} \big(F(t) - \zeta(t)\big)^2 + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0. \tag{2.18}$$

Using the essential inequality $(x+y)^2 \geq \frac{1}{\alpha+1}x^2 - \frac{1}{\alpha}y^2$ holding for all real numbers x, y and positive real number α, we get

$$\big(F(t) - \zeta(t)\big)^2 \geq \frac{1}{\alpha+1} F^2(t) - \frac{1}{\alpha} \zeta^2(t). \tag{2.19}$$

Substituting (2.19) into (2.18) and taking $\alpha = \frac{1}{n-1} > 0$, we have

$$\operatorname{Ric}(\gamma'(t), \gamma'(t)) \leq -F'(t) + \zeta'(t) - \frac{1}{n}F^2(t) + \zeta^2(t). \tag{2.20}$$

If we add $(\operatorname{Hess} f)(\gamma'(t), \gamma'(t))$ to the both sides of inequality (2.20), we have

$$\operatorname{Ric}_f(\gamma'(t), \gamma'(t)) \leq -F'(t) + \zeta'(t) - \frac{1}{n}F^2(t) + \zeta^2(t) + (\operatorname{Hess} f)(\gamma'(t), \gamma'(t)). \tag{2.21}$$

Integrating both sides of the inequality (2.21) from 1 to t, we obtain

$$\begin{aligned}\int_1^t \operatorname{Ric}_f(\gamma'(s), \gamma'(s))ds \leq & -F(t) + F(1) - \int_1^t \frac{1}{n}F^2(s)ds + \int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds \\ & + g(\nabla f, \gamma')(t) - g(\nabla f, \gamma')(1).\end{aligned} \tag{2.22}$$

Therefore, under the assumption

$$\int_0^\infty \operatorname{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty \tag{2.23}$$

given in Theorem 1.5, we have

$$\lim_{t\to\infty} -F(t) - \int_1^t \frac{1}{n}F^2(s)ds + \int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds + f'(t) = \infty, \tag{2.24}$$

where $f' = \frac{d}{dt}f(\gamma(t)) = g(\nabla f, \gamma')$. Here, multiplying by $1/n$ on both sides then yields

$$\lim_{t\to\infty} -\frac{1}{n}F(t) - \int_1^t \Big(\frac{1}{n}F(s)\Big)^2 ds + \frac{1}{n}\int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds + \frac{1}{n}f'(t) = \infty. \tag{2.25}$$

Because of (2.24), given $C > 1$ there exists $t_1 > 1$ such that

$$-\frac{1}{n}F(t) - \int_1^t \Big(\frac{1}{n}F(s)\Big)^2 ds + \frac{1}{n}\int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds + \frac{1}{n}f'(t) \geq C \tag{2.26}$$

for all $t \geq t_1$.

On the other hand, under the assumption $f'(t) \leq \frac{1}{4}(1 - \frac{1}{t})$ of Theorem 1.5, if the function ζ is taken to be $\zeta(t) = \frac{1}{2t}$, then we get the following inequality

$$-\frac{1}{n}F(t) - \int_1^t \Big(\frac{1}{n}F(s)\Big)^2 ds \geq C \tag{2.27}$$

for all $t \geq t_1$.

Let us now consider an increasing sequence $\{t_\ell\}$ defined by

$$t_{\ell+1} = t_\ell + C^{1-\ell}, \quad \text{for } \ell \geq 1, \tag{2.28}$$

such that $\{t_\ell\}$ converges to $T := t_1 + \frac{C}{C-1}$ as $\ell \to \infty$.

We claim the fact that $-F(t) \geq nC^\ell$ for all $t \geq t_\ell$: To prove the claim, we use induction argument. It is trivial from inequality (2.27) for $\ell = 1$. By induction, we get the claim for ℓ.

Then we must prove that $-F(t) \geq nC^{\ell+1}$ for all $t \geq t_{\ell+1}$. By means of the inequality (2.27), we obtain

$$\begin{aligned} -F(t) &\geq nC + \frac{1}{n}\int_1^t F^2(s)ds \\ &\geq \frac{1}{n}\int_1^{t_\ell} F^2(s)ds + \frac{1}{n}\int_{t_\ell}^t F^2(s)ds \\ &\geq \frac{1}{n}\int_{t_\ell}^t F^2(s)ds \\ &\geq nC^{2\ell}(t-t_\ell) \\ &\geq nC^{2\ell}(t_{\ell+1}-t_\ell) = nC^{\ell+1}. \end{aligned} \tag{2.29}$$

This proves the above claim.

From hence, we have

$$\lim_{\ell\to\infty} -F(t_\ell) = -F(T) \geq \lim_{\ell\to\infty} nC^\ell. \tag{2.30}$$

However, this result contradicts with the smoothness of $F(t)$. Namely, $\lim_{t\to T^-} -F(t) = \infty$. This completes the proof of Theorem 1.5. ∎

On the other hand, under the same assumptions given in the Theorem 1.4, we see that, the above diameter estimate given by (1.6) can be improved as follows:

Proof of Theorem 1.6. Let $p, q \in M$ be two distinct point and γ a minimizing unit speed geodesic segment from p to q of length $\ell > 0$. Let $\{E_1 = \gamma', E_2, \ldots, E_n\}$ be a parallel orthonormal frame along γ and let $h \in \mathcal{C}^\infty([0,\ell])$ be a real-valued smooth function such that $h(0) = h(\ell) = 0$. Then, from the index form of γ, we have

$$\sum_{i=2}^n \mathrm{I}(hE_i, hE_i) = \int_0^\ell \Big((n-1)h'^2 - h^2\mathrm{Ric}(\gamma',\gamma')\Big)dt. \tag{2.31}$$

Using the assumption (1.8) given in Theorem 1.6 in the integral expression (2.31), we get

$$\sum_{i=2}^n \mathrm{I}(hE_i, hE_i) \quad \leq \quad \int_0^\ell \Big((n-1)h'^2 - ah^2 - h^2\frac{d\phi}{dt}\Big)dt. \tag{2.32}$$

In the inequality (2.32), the term $-h^2\frac{d\phi}{dt}$ equals to

$$-h^2\frac{d\phi}{dt} = -\frac{d}{dt}(h^2\phi) + 2hh'\phi. \tag{2.33}$$

Integrating both sides of (2.33), we get

$$\int_0^\ell -h^2\frac{d\phi}{dt}dt = 2\int_0^\ell hh'\phi dt \leq 2\int_0^\ell \big|hh'\phi\big|dt \leq 2c\int_0^\ell \big|hh'\big|dt. \tag{2.34}$$

Thus, under the choice $h(t) = \sin(\frac{\pi t}{\ell})$, we have

$$\sum_{i=2}^{n} \mathrm{I}(hE_i, hE_i) \leq \frac{1}{2\ell}\left[(n-1)\pi^2 - a\ell^2 + 4c\ell\right]. \tag{2.35}$$

Since γ is a minimal geodesic, we must take

$$a\ell^2 - 4c\ell - (n-1)\pi^2 \leq 0. \tag{2.36}$$

This inequality gives

$$\ell \leq \frac{1}{a}\Big(2c + \sqrt{4c^2 + a(n-1)\pi^2}\Big). \tag{2.37}$$

This completes the proof of Theorem 1.6. $\square$

References

[1] Ambrose W. A theorem of Myers, Duke Math J 1957; 24: 345-348.

[2] Bakry D, Émery M. Diffusions hypercontractives, In: Séminaire de Probabilités XIX, Lect Notes in Math 1985; 1123: 177-206.

[3] Cavalcante MP, Oliveira JQ, Santos MS. Compactness in weighted manifolds and applications, Results Math 2015; 68(1): 143-156.

[4] Cheeger J, Gromov M, Taylor M. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J Differ Geom 1982; 17: 15-53.

[5] Fernández-López M, García-Río E. A remark on compact Ricci solitons, Math Ann 2008; 340: 893-896.

[6] Galloway GJ. A generalization of Myers theorem and an application to relativistic cosmology, J Differ Geom 1979; 14(1): 105-116.

[7] Kuwada K. A probabilistic approach to the maximal diameter theorem, Math Nachr 2013; 286(4): 374–378.

[8] Lee JM. Riemannian Manifolds: An Introduction to Curvature, New York: Springer-Verlag, 1997.

[9] Limoncu M. Modifications of the Ricci tensor and applications, Arch Math 2010; 95(2): 191-199.

[10] Limoncu M. The Bakry-Emery Ricci tensor and its applications to some compactness theorems, Math Z 2012; 271: 715-722.

[11] Lott J. Some geometric properties of the Bakry-Emery-Ricci tensor, Comment Math Helv 2003; 78: 865-883.

[12] Mastrolia P, Rimoldi M, Veronelli G. Myers' type theorems and some related oscillation results, J Geom Anal 2012; 22: 763–779.

[13] Myers SB. Riemannian manifolds with positive mean curvature, Duke Math J 1941; 8: 401-404.

[14] O'Neill B. Semi-Riemannian Geometry with Applications to Relativity, London: Academic Press, 1983.

[15] Soylu Y. A Myers-type compactness theorem by the use of Bakry-Emery Ricci tensor, Diff Geom Appl 2017; 54: 245-250.

[16] Tadano H. Remark on a diameter bound for complete Riemannian manifolds with positive Bakry-Émery Ricci curvature, Differ Geom Appl 2016; 44: 136-143.

[17] Wang LF. A Myers theorem via m-Bakry-Émery curvature, Kodai Math J 2014; 37: 87-195.

[18] Wei G, Wylie W. Comparison geometry for the Bakry-Emery Ricci tensor, J Differ Geom 2009; 83: 377-405.

[19] Zhang S. A theorem of Ambrose for Bakry-Emery Ricci tensor, Ann Glob Anal Geom 2014; 45: 233-238.

Giresun University, Department of Mathematics, 28100, Giresun-Turkey

Email address: yasemin.soylu@giresun.edu.tr

International Journal of Maps in Mathematics

Volume:1, Issue:2, (2018), Pages:162-174

ISSN: 2636-7467 (Online)

www.journalmim.com

ON GENERALIZED SASAKIAN SPACE FORMS WITH CONCIRCULAR AND PROJECTIVE CURVATURE TENSOR

GURUPADAVVA INGALAHALLI* AND C.S. BAGEWADI

Abstract. In this paper we study the Concircular pseudosymmetric, $\tilde{C}(\xi, X) \cdot R = 0$, $\tilde{C} \cdot Q = 0$, $Q \cdot \tilde{C} = 0$, Projective pseudosymmetric, $P(\xi, X) \cdot R = 0$, $P \cdot Q = 0$ and $Q \cdot P = 0$ in generalized Sasakian space forms.

1. Introduction

Alegre P, Blair DE, Carriazo A. [1] introduced and studied the concept of generalized Sasakian space forms. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a generalized Sasakian space form if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of M is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

Received:2018-03-04 *Revised: 2018-05-12* *Accepted:2018-09-08*

*2010 Mathematics Subject Classification.*53C15, 53C25, 53D15.

Key words: Generalized Sasakian space form, Concircular curvature tensor, projective curvature tensor, Ricci tensor, Ricci operator, scalar curvature.

* Corresponding author

for any vector fields X, Y, Z on M. Throughout the paper we denote generalized Sasakian space form as $M(f_1, f_2, f_3)$, which appears as a natural generalization of the Sasakian space form $M(c)$, which can be obtained as a particular case of generalized Sasakian space form by taking $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. The notion of generalized Sasakian space forms have been weakened by many geometers such as [2, 3, 4, 5, 8, 9, 14, 15, 17, 19] with different curvature tensors.

A Riemannian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). As a proper generalization of locally symmetric manifold, the notion of semi-symmetric manifold was defined by $(R(X,Y) \cdot R)(U,V)W = 0$.

For a $(0,k)$-tensor field T on $M, k \geq 1$, and a symmetric $(0,2)$-tensor field g on M, we define the tensor fields $R \cdot T$ and $Q(g,T)$ by

$$(R \cdot T)(X_1, \ldots, X_k; X, Y) = -T(R(X,Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots X_{k-1}, R(X,Y)X_k)$$

and

$$Q(g,T)(X_1, \ldots, X_k; X, Y) = -T((X \wedge_g Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots X_{k-1}, (X \wedge_g Y)X_k).$$

Where $X \wedge_g Y$ is the endomorphism given by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{1.1}$$

A Riemannian manifold M is said to be pseudosymmetric [11] if

$$R \cdot R = L_R Q(g,R) \tag{1.2}$$

holds on $U_R = \{x \in M | R - \frac{r}{n(n-1)} G \neq 0 \ at \ x\}$, where G is the $(0,4)$-tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and L_R is some smooth function on M. A Riemannian manifold M is said to be Concircular pseudosymmetric if

$$R \cdot \tilde{C} = L_{\tilde{C}} Q(g, \tilde{C}) \tag{1.3}$$

holds on the set $U_{\tilde{C}} = \{x \in M : \tilde{C} \neq 0\}$ at x, where $L_{\tilde{C}}$ is some function on $U_{\tilde{C}}$ and $\tilde{C}$ is the Concircular curvature tensor. It is known that every pseudosymmetric manifold is Concircular pseudosymmertic, but the converse is not true. If $L_{\tilde{C}} = 0$ on $U_{\tilde{C}}$, then a Concircular pseudosymmetric manifold is Concircular semisymmetric. But $L_{\tilde{C}}$ need not be zero, in general and hence there exists Concircular pseudosymmetric manifolds which are not Concircular semisymmetric. Thus the class of Concircular pseudosymmetric manifolds is a natural extension of the class of Concircular semisymmetric manifolds.

Motivated by the above work in this paper we study the Concircular pseudosymmetric, $\tilde{C}(\xi, X) \cdot R = 0$, $\tilde{C} \cdot Q = 0$, $Q \cdot \tilde{C} = 0$, Projective pseudosymmetric, $P(\xi, X) \cdot R = 0$, $P \cdot Q = 0$ and $Q \cdot P = 0$ in generalized Sasakian space forms.

2. Preliminaries

An n-dimensional Riemannian manifold M is called an almost contact metric manifold [7] if there exist a $(1, 1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.5}$$

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.6}$$

For an n-dimensional generalized Sasakian space form [1], we have

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ &+ 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &- g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{2.7}$$

$$S(X, Y) = [(n-1)f_1 + 3f_2 - f_3]g(X, Y) + [-3f_2 - (n-2)f_3]\eta(X)\eta(Y), \tag{2.8}$$

$$r = (n-1)\{nf_1 + 3f_2 - 2f_3\}. \tag{2.9}$$

From (2.7) and (2.8), we get

$$\eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{2.10}$$

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.11}$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.12}$$

$$S(X, \xi) = (n-1)(f_1 - f_3)\eta(X), \tag{2.13}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor and r is the Scalar curvature.

3. **Concircular Pseudosymmetric Generalized Sasakian Space Forms**

This section deals with the study of Concircular pseudosymmetric generalized Sasakian space forms. A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle is called a concircular transformation ([13], [21]). A concircular transformation is always a conformal transformation ([13]). Here

geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by ([21])

$$\tilde{C}(X,Y)Z \quad = \quad R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.14}$$

where R is the curvature tensor and r is the scalar curvature of the manifold.

Let $M(f_1, f_2, f_3)$ be an n-dimensional Concircular pseudosymmetric generalized Sasakian space form. Then from (1.3), we have

$$(R(\xi,Y)\cdot\tilde{C})(U,V)W = L_{\tilde{C}}[(\xi\wedge Y)\cdot\tilde{C}(U,V)W]. \tag{3.15}$$

By (3.15), we get

$$\begin{aligned}
&R(\xi,Y)\tilde{C}(U,V)W - \tilde{C}(R(\xi,Y)U,V)W - \tilde{C}(U,R(\xi,Y)V)W - \tilde{C}(U,V)R(\xi,Y)W\\
&= L_{\tilde{C}}[(\xi\wedge Y)\tilde{C}(U,V)W - \tilde{C}((\xi\wedge Y)U,V)W - \tilde{C}(U,(\xi\wedge Y)V)W\\
&-\tilde{C}(U,V)(\xi\wedge Y)W].
\end{aligned} \tag{3.16}$$

By using the expression (2.12) in (3.16), we have

$$\begin{aligned}
&(L_{\tilde{C}} - (f_1 - f_3))[g(Y,\tilde{C}(U,V)W)\xi - \eta(\tilde{C}(U,V)W)Y - g(Y,U)\tilde{C}(\xi,V)W\\
&+\eta(U)\tilde{C}(Y,V)W - g(Y,V)\tilde{C}(U,\xi)W + \eta(V)\tilde{C}(U,Y)W\\
&-g(Y,W)\tilde{C}(U,V)\xi + \eta(W)\tilde{C}(U,V)Y] = 0.
\end{aligned} \tag{3.17}$$

By taking the inner product with ξ in (3.17), we obtain

$$\begin{aligned}
&(L_{\tilde{C}} - (f_1 - f_3))[g(Y,\tilde{C}(U,V)W) - \eta(\tilde{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\tilde{C}(\xi,V)W)\\
&+\eta(U)\eta(\tilde{C}(Y,V)W) - g(Y,V)\eta(\tilde{C}(U,\xi)W) + \eta(V)\eta(\tilde{C}(U,Y)W)\\
&-g(Y,W)\eta(\tilde{C}(U,V)\xi) + \eta(W)\eta(\tilde{C}(U,V)Y)] = 0.
\end{aligned} \tag{3.18}$$

By (3.18), we get either $L_{\tilde{C}} = (f_1 - f_3)$ or

$$\begin{aligned}
&[g(Y,\tilde{C}(U,V)W) - \eta(\tilde{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\tilde{C}(\xi,V)W)\\
&+\eta(U)\eta(\tilde{C}(Y,V)W) - g(Y,V)\eta(\tilde{C}(U,\xi)W) + \eta(V)\eta(\tilde{C}(U,Y)W)\\
&-g(Y,W)\eta(\tilde{C}(U,V)\xi) + \eta(W)\eta(\tilde{C}(U,V)Y)] = 0.
\end{aligned} \tag{3.19}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold. Putting $U = Y = e_i$ in (3.19) and taking summation over $i, (1 \leq i \leq n)$ and by virtue of (3.14), we have

$$S(V, W) = (n-1)(f_1 - f_3)g(V, W). \tag{3.20}$$

On contracting (3.20), we get

$$r = n(n-1)(f_1 - f_3). \tag{3.21}$$

Therefore, $M(f_1, f_2, f_3)$ is an Einstein manifold. Hence we state the following theorem.

Theorem 3.1. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form. If $M(f_1, f_2, f_3)$ is Concircular pseudosymmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold or $L_{\tilde{C}} = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

Now, by using (3.21) in (3.14) then we get

$$\eta(\tilde{C}(X, Y)Z) = 0 \tag{3.22}$$

and

$$\eta(\tilde{C}(\xi, Y)Z) = 0. \tag{3.23}$$

By virtue of (3.22) and (3.23) in (3.19), we obtain

$$g(Y, \tilde{C}(U, V)W) = \tilde{C}(U, V, W, Y) = 0. \tag{3.24}$$

This implies that $M(f_1, f_2, f_3)$ is Concircularly flat. Hence we conclude the following theorem.

Theorem 3.2. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form. If $M(f_1, f_2, f_3)$ is Concircular pseudosymmetric then $M(f_1, f_2, f_3)$ is either Concircularly flat or $L_{\tilde{C}} = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

If we assume that $M(f_1, f_2, f_3)$ is not Concircularly semi symmtric, a Concircular pseudosymmetric generalized Sasakian space form. Then we get $R \cdot \tilde{C} = (f_1 - f_3)Q(g, \tilde{C})$, which implies that the pseudosymmetry function $L_{\tilde{C}} = (f_1 - f_3)$. Therefore we have the following:

Corollary 3.1. *Every generalized Sasakian space form $M(f_1, f_2, f_3)$ is Concircular pseudosymmetric of the form $R \cdot \tilde{C} = (f_1 - f_3)Q(g, \tilde{C})$.*

4. Generalized Sasakian space form satisfying $\tilde{C}(\xi, X) \cdot R = 0$

In this section we study generalized Sasakian space form satisfying $\tilde{C}(\xi, X) \cdot R = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $\tilde{C}(\xi, X) \cdot R = 0$. Then, we have

$$\begin{aligned}(\tilde{C}(\xi, X) \cdot R)(U, V)W &= \tilde{C}(\xi, X)R(U, V)W - R(\tilde{C}(\xi, X)U, V)W \\ &- R(U, \tilde{C}(\xi, X)V)W - R(U, V)\tilde{C}(\xi, X)W = 0. \qquad (4.25)\end{aligned}$$

Putting $W = \xi$ in (4.25) and by virtue of (2.11), we obtain

$$\begin{aligned}&(f_1 - f_3)\eta(\tilde{C}(\xi, X)U)V - (f_1 - f_3)\eta(\tilde{C}(\xi, X)V)U \\ &-[(f_1 - f_3) - \frac{r}{n(n-1)}]\{\eta(X)R(U, V)\xi - R(U, V)X\} = 0. \qquad (4.26)\end{aligned}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking inner product with e_i in (4.26) and on simplification, we get

$$S(X, V) = (n-1)(f_1 - f_3)g(X, V). \qquad (4.27)$$

On Contracting (4.27), we have

$$r = n(n-1)(f_1 - f_3). \qquad (4.28)$$

Conversely, if $f_1 = f_3$ then from (2.12) and (3.14) trivially we get $\tilde{C}(\xi, X) \cdot R = 0$. If $S(X, V) = (n-1)(f_1 - f_3)g(X, V)$ with scalar curvature $r = n(n-1)(f_1 - f_3)$, we obtain $\tilde{C}(\xi, X) \cdot R = 0$. And then comparing r with (2.9) we have $3f_2 + (n-2)f_3 = 0$. Hence we conclude the following theorem.

Theorem 4.1. *An n-dimensional generalized Sasakian space form M satisfying the condition $\tilde{C}(\xi, X) \cdot R = 0$ if and only if either $f_1 = f_3$ or the manifold is an Einstein manifold with scalar curvature $r = n(n-1)(f_1 - f_3)$.*

Remark 4.1. *In [4], author obtained necessary and sufficient condition for a generalized Sasakian space form M^{2n+1} satisfying $\tilde{C}(\xi, X) \cdot R = 0$ if and only if the functions f_2 and f_3 either satisfy the conditions $(2n-1)f_3 + 3f_2 = 0$ or it has the sectional curvature $(f_1 - f_3)$.*

5. Generalized Sasakian space form satisfying $\tilde{C} \cdot Q = 0$

In this section we study the generalized Sasakian space form satisfying $\tilde{C} \cdot Q = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $\tilde{C} \cdot Q = 0$. Then, we have

$$\tilde{C}(X,Y)QZ - Q(\tilde{C}(X,Y)Z) = 0, \tag{5.29}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (5.29), we have

$$\tilde{C}(X,\xi)QZ - Q(\tilde{C}(X,\xi)Z) = 0. \tag{5.30}$$

By using (3.14) in (5.30) and on simplification, we obtain

$$\begin{aligned}\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right] \quad & [(n-1)(f_1 - f_3)\eta(Z)X - S(X,Z)\xi \\ & -\eta(Z)QX + (n-1)(f_1 - f_3)g(X,Z)\xi] = 0.\end{aligned} \tag{5.31}$$

Taking inner product with ξ in (5.31), we have

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][(n-1)(f_1 - f_3)g(X,Z) - S(X,Z)] = 0. \tag{5.32}$$

From (5.32), either $[(f_1 - f_3) - \frac{r}{n(n-1)}] = 0$ or

$$S(X,Z) = (n-1)(f_1 - f_3)g(X,Z). \tag{5.33}$$

Hence, we state the following theorem.

Theorem 5.1. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfies the curvature condition $\tilde{C} \cdot Q = 0$, then the manifold is an Einstein manifold or the scalar curvature $r = n(n-1)(f_1 - f_3)$.*

6. Generalized Sasakian space form satisfying $Q \cdot \tilde{C} = 0$

In this section we study generalized Sasakian space form satisfying $Q \cdot \tilde{C} = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $Q \cdot \tilde{C} = 0$. Then, we have

$$Q(\tilde{C}(X,Y)Z) - \tilde{C}(QX,Y)Z - \tilde{C}(X,QY)Z - \tilde{C}(X,Y)QZ = 0, \tag{6.34}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (6.34), we have

$$Q(\tilde{C}(X,\xi)Z) - \tilde{C}(QX,\xi)Z - \tilde{C}(X,Q\xi)Z - \tilde{C}(X,\xi)QZ = 0. \tag{6.35}$$

By using (3.14) in (6.35) and on simplification, we obtain

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][2S(X,Z)\xi - 2(n-1)(f_1 - f_3)\eta(Z)X] = 0. \tag{6.36}$$

Taking inner product with ξ in (6.36), we have

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][2S(X,Z) - 2(n-1)(f_1 - f_3)\eta(Z)\eta(X)] = 0. \tag{6.37}$$

Putting $Z = \xi$ in (6.37), then from (6.37) either $[(f_1 - f_3) - \frac{r}{n(n-1)}] = 0$ or

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \tag{6.38}$$

which implies

$$Q\xi = (n-1)(f_1 - f_3)\xi. \tag{6.39}$$

Hence, we state the following theorem.

Theorem 6.1. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfies the curvature condition $Q \cdot \tilde{C} = 0$, then the Ricci operator of ξ of a generalized Sasakian space form is equal to $(n-1)$ times of $(f_1 - f_3)\xi$ or the scalar curvature $r = n(n-1)(f_1 - f_3)$.*

7. Projective Pseudosymmetric Generalized Sasakian Space Forms

This section deals with the study of Projective pseudosymmetric generalized Sasakian space forms. The projective curvature tensor is an important concept of Riemannian geometry, which one uses to calculate the basic geometric measurements on a manifold. The projective transformation on a manifold is a transformation under which geodesic transforms into geodesic. The projective curvature tensor is given by ([4])

$$P(X,Y)Z \quad = \quad R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{7.40}$$

Let $M(f_1, f_2, f_3)$ be an n-dimensional Projective pseudosymmetric generalized Sasakian space form. Then we have

$$(R(\xi,Y)\cdot P)(U,V)W = L_P[(\xi\wedge Y)\cdot P(U,V)W]. \tag{7.41}$$

By (7.41), we get

$$\begin{aligned}
&R(\xi,Y)P(U,V)W - P(R(\xi,Y)U,V)W - P(U,R(\xi,Y)V)W - P(U,V)R(\xi,Y)W \\
&= L_P[(\xi\wedge Y)P(U,V)W - P((\xi\wedge Y)U,V)W - P(U,(\xi\wedge Y)V)W \\
&-P(U,V)(\xi\wedge Y)W].
\end{aligned} \tag{7.42}$$

By using the expression (2.12) in (7.42), we have

$$(L_P-(f_1-f_3))[g(Y,P(U,V)W)\xi-\eta(P(U,V)W)Y-g(Y,U)P(\xi,V)W$$
$$+\eta(U)P(Y,V)W-g(Y,V)P(U,\xi)W+\eta(V)P(U,Y)W$$
$$-g(Y,W)P(U,V)\xi+\eta(W)P(U,V)Y]=0. \quad (7.43)$$

By taking the inner product with ξ in (7.43), we obtain

$$(L_P-(f_1-f_3))[g(Y,P(U,V)W)-\eta(P(U,V)W)\eta(Y)-g(Y,U)\eta(P(\xi,V)W)$$
$$+\eta(U)\eta(P(Y,V)W)-g(Y,V)\eta(P(U,\xi)W)+\eta(V)\eta(P(U,Y)W)$$
$$-g(Y,W)\eta(P(U,V)\xi)+\eta(W)\eta(P(U,V)Y)]=0. \quad (7.44)$$

By (7.44), we get either $L_P=(f_1-f_3)$ or

$$[g(Y,P(U,V)W)-\eta(P(U,V)W)\eta(Y)-g(Y,U)\eta(P(\xi,V)W)$$
$$+\eta(U)\eta(P(Y,V)W)-g(Y,V)\eta(P(U,\xi)W)+\eta(V)\eta(P(U,Y)W)$$
$$-g(Y,W)\eta(P(U,V)\xi)+\eta(W)\eta(P(U,V)Y)]=0. \quad (7.45)$$

Let $\{e_1,e_2,\dots,e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold. Putting $U=Y=e_i$ in (7.45) and taking summation over $i,(1\le i\le n)$ and by virtue of (7.40), we have

$$S(V,W)=(n-1)(f_1-f_3)g(V,W)-\left[\frac{r}{n-1}-n(f_1-f_3)\right]\eta(V)\eta(W). \quad (7.46)$$

On contracting (7.46), we get

$$r=n(n-1)(f_1-f_3). \quad (7.47)$$

By using (7.47) in (7.46), we obtain

$$S(V,W)=(n-1)(f_1-f_3)g(V,W). \quad (7.48)$$

Therefore, $M(f_1,f_2,f_3)$ is an Einstein manifold. Hence, we state the following theorem.

Theorem 7.1. *Let $M(f_1,f_2,f_3)$ be an n-dimensional generalized Sasakian space form. If $M(f_1,f_2,f_3)$ is Projective pseudosymmetric then $M(f_1,f_2,f_3)$ is an Einstein manifold or $L_P=(f_1-f_3)$ holds on $M(f_1,f_2,f_3)$.*

Now, by using (2.10) and (7.48) in (7.40) then we get

$$\eta(P(X,Y)Z) = 0 \tag{7.49}$$

and

$$\eta(P(\xi,Y)Z) = 0. \tag{7.50}$$

By virtue of (7.49) and (7.50) in (7.45), we obtain

$$g(Y, P(U,V)W) = P(U,V,W,Y) = 0. \tag{7.51}$$

This implies that $M(f_1, f_2, f_3)$ is Projectively flat. Hence, we conclude the following theorem.

Theorem 7.2. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form. If $M(f_1, f_2, f_3)$ is Projective pseudosymmetric then $M(f_1, f_2, f_3)$ is either Projectively flat or $L_P = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

If we assume that $M(f_1, f_2, f_3)$ is not Projectively semisymmtric, a Projective pseudosymmetric generalized Sasakian space form. Then we get $R \cdot P = (f_1 - f_3)Q(g, P)$, which implies that the pseudosymmetry function $L_P = (f_1 - f_3)$. Therefore we have the following:

Corollary 7.1. *Every generalized Sasakian space form $M(f_1, f_2, f_3)$ is Projective pseudosymmetric of the form $R \cdot P = (f_1 - f_3)Q(g, P)$.*

8. Generalized Sasakian space form satisfying $P(\xi, X) \cdot R = 0$

In this section we study generalized Sasakian space form satisfying $P(\xi, X) \cdot R = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $P(\xi, X) \cdot R = 0$. Then, we have

$$\begin{aligned} &P(\xi,X)R(U,V)W - R(P(\xi,X)U,V)W \\ &-R(U,P(\xi,X)V)W - R(U,V)P(\xi,X)W = 0. \end{aligned} \tag{8.52}$$

Putting $W = \xi$ in (8.52) and by virtue of (2.11), we obtain

$$\begin{aligned} &(f_1 - f_3)\eta(P(\xi,X)U)V - (f_1 - f_3)\eta(P(\xi,X)V)U \\ &-(n-2)(f_1 - f_3)\{\eta(X)R(U,V)\xi - R(U,V)X\} = 0. \end{aligned} \tag{8.53}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking inner product with e_i in (8.53) and on simplification, we get

$$S(X,V) = (n-1)(f_1 - f_3)g(X,V) + (n-2)(f_1 - f_3)\eta(X)\eta(V). \tag{8.54}$$

Therefore, $M(f_1, f_2, f_3)$ is an η-Einstein manifold. Hence, we state the following theorem.

Theorem 8.1. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfying the condition $P(\xi, X) \cdot R = 0$ is an η-Einstein manifold.*

9. Generalized Sasakian space form satisfying $P \cdot Q = 0$

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $P \cdot Q = 0$. Then, we have

$$P(X, Y)QZ - Q(P(X, Y)Z) = 0, \tag{9.55}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (9.55), we have

$$P(X, \xi)QZ - Q(P(X, \xi)Z) = 0. \tag{9.56}$$

By using (7.40), (2.11) in (9.56) and on simplification, we obtain

$$\frac{1}{(n-1)} S(X, QZ)\xi - 2(f_1 - f_3)S(X, Z)\xi + (n-1)(f_1 - f_3)^2 g(X, Z)\xi = 0. \tag{9.57}$$

Taking inner product with ξ in (9.57), we have

$$S^2(X, Z) = 2(n-1)(f_1 - f_3)S(X, Z) - (n-1)^2 g(f_1 - f_3)^2 g(X, Z). \tag{9.58}$$

Hence, we state the following theorem.

Theorem 9.1. *An n-dimensional generalized Sasakian space form satisfies the curvature condition $P \cdot Q = 0$, then the square of the Ricci tensor S^2 is the linear combination of the Ricci tensor S and the metric tensor g.*

10. Generalized Sasakian space form satisfying $Q \cdot P = 0$

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $Q \cdot P = 0$. Then, we have

$$Q(P(X, Y)Z) - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0, \tag{10.59}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (10.59), we have

$$Q(P(X, \xi)Z) - P(QX, \xi)Z - P(X, Q\xi)Z - P(X, \xi)QZ = 0. \tag{10.60}$$

By virtue of (7.40) in (10.60) and on simplification, we obtain

$$2(f_1 - f_3)S(X, Z)\xi - \frac{2}{(n-1)} S(X, QZ)\xi = 0. \tag{10.61}$$

Taking inner product with ξ in (10.61), we have

$$S(X, QZ) = (n-1)(f_1 - f_3)S(X, Z), \tag{10.62}$$

which implies

$$g(Q^2X, Z) = (n-1)(f_1 - f_3)g(QX, Z), \tag{10.63}$$

Let $\{e_1, e_2, e_3, \ldots, e_n\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = Z = e_i$ in (10.63) and taking summation over i, we have

$$trace(Q^2) = (n-1)(f_1 - f_3)trace(Q). \tag{10.64}$$

Hence, we state the following theorem.

Theorem 10.1. *An n-dimensional generalized Sasakian space form satisfies the curvature condition $Q \cdot P = 0$, then the trace of the square Ricci operator of a generalized Sasakian space form is equal to $(n-1)(f_1 - f_3)$ times of trace of the Ricci operator.*

Acknowledgement

The authors are thankful to the referees for their comments and valuable suggestions towards the improvement of this paper.

References

[1] Alegre P, Blair DE, Carriazo A. Generalized Sasakian-space-form, Israel J Math 2004; 14: 157-183.

[2] Alegre P, Carriazo A. Structures on Generalized Sasakian-space-form, Diff Geo and its Application 2008; 26(6): 656-666.

[3] Alegre P, Carriazo A. Generalized Sasakian-space-forms and Conformal Changes of the Metric, Results Math 2011; 59: 485-493.

[4] Atcecken A. On generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor, Bulletin of Mathematics Analysis and Applications 2014; 6(1): 1-8.

[5] Bagewadi CS, Gurupadavva Ingalahalli. A Study on Curvature tensors of a Generalized Sasakian space form, Acta Universitatis Apulensis 2014; 38: 81-93.

[6] Belkhelfa M, Deszcz R, Verstraelen L. Symmetry Properties of Sasakian-space-form, Soochow Journal of Mathematics 2005; 31(4): 19-36.

[7] Blair DE. Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics, Springer-Verlag, Berlin 1976; 509.

[8] Carriazo A, Blair DE, Alegre P. On generalized Sasakian-space-form, Proceedings of the Ninth International Workshop on Diff Geom 2005; 9: 31-39.

[9] De UC, Haseeb A. On generalized Sasakian-space-forms with M-projective curvature tensor, Adv. Pure Appl. Math. 2018; 9(1): 67-73.

[10] De UC, Sarkar A. On the Projective Curvature Tensor of Generalized Sasakian-Space-Forms, Quaestiones Mathematicae 2010; 33(2): 245-252.

[11] Deszcz R. On pseudosymmetric spaces, Bull Soc Math Belg Ser A 1992; 44(1): 1-34.

[12] Kim UK. Conformally flat Generalized Sasakian-space-forms and locally symmetric Generalized Sasakian-space-form, Note di Matematica 2006; 26(1): 55-67.

[13] Kuhnel W. Conformal transformations between Einstein spaces, Conformal geometry (Bonn, 1985/1986), 105-146, Aspects Math., E12, Vieweg, Braunschweig, 1988.

[14] Hui SK, Chakraborty D. Generalized Sasakian-space-forms and Ricci almost solitons with a conformal killing vector field, New Trends in Math Sciences 2016; 4: 263-269.

[15] Hui SK, Prakasha DG. On the C-Bochner curvature tensor of generalized Sasakian-space-forms, Proc Natl Acad Sci India Sect A phys Sci 2015; 85(3): 401-405.

[16] Hui SK, Lemence RS, Chakraborty D. Ricci solitons on Ricci Pserdosymmetric $(LCS)_n$-Manifolds, arXiv:1707.03618v1 [math.DG] 12 Jul 2017.

[17] Nagaraja HG, Somashekhara G, Savithri Shashidhar. On Generalized Sasakian-Space-Forms, ISRN Geometry 2012; 1-12.

[18] Ozgur C, Tripathi MM. On P-Sasakian manifods satisfying certain conditions on concircular curvature tensor, Turk J Math 2007; 31: 171-179.

[19] Prakasha DG. On Generalized Sasakian-space-forms with Weyl-Conformal Curvature tensor, Lobachevskii Journal of Mathematics 2012; 33(3): 223-228.

[20] Majhi P, Ghosh G. On a Classification of Para-Sasakian Manifolds, Facta Universitatis Ser Math Inform 2017; 32(5): 781-788.

[21] Yano K. Concircular geometry I. Concircular transformations, Proc Imp Acad Tokyo 1940; 16: 195-200.

Department of Mathematics, J.N.N.C.E, Shimoga - 577 204, Karnataka, INDIA.

Email address: gurupadavva@gmail.com

Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA.

Email address: prof_bagewadi@yahoo.co.in

International Journal of Maps in Mathematics

Volume:1, Issue:2, (2018), Pages:175-187

ISSN: 2636-7467 (Online)

www.journalmim.com

EXISTENCE AND STABILITY FOR A LAMÉ SYSTEM WITH TIME DELAY AND INFINITE MEMORY

A. BENIANI, KH. ZENNIR*, A. BENAISSA, AND S. BOULAARES

Abstract. We pursue the investigation started in a recent paper [10] and later [2] concerning the wave equations with elasticity operator. We prove the existence of solutions for Lamé system in three-dimension bounded domain with time delay term by using semi-group theory. We also study the exponential stability of solutions by means of an appropriate Lyapunov functional.

1. Introduction and related results

Let us define the elasticity operator Δ_e, which is the 3×3 matrix-valued differential operator by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

where μ, λ are the Lamé constants which satisfy

$$\begin{aligned} &\mu > 0, \\ &\lambda + \mu \geq 0. \end{aligned} \tag{1.1}$$

It is well known that for the case $\lambda + \mu = 0$, $\Delta_e = \mu\Delta$ gives the Laplacian operator.

Received:2018-03-08 *Accepted:2018-05-09*

2010 Mathematics Subject Classification: 35L05, 35L15, 35L70, 35B40

Key words: Lamé system, Delay term, Lyapunov functions, Decay rates

* Corresponding author

In this paper, we consider the following Lamé system with time delay term and infinite memory:

$$u''(x,t)-\Delta_e u(x,t)+\int_0^{+\infty}h(s)\Delta u(x,t-s)ds+ku'(x,t-\tau)=0 \quad \text{in } \Omega\times\mathbb{R}^+. \tag{1.2}$$

Eq.(1.2) supplemented with initial and boundary conditions

$$\begin{cases} u(x,-t)=u_0(x,t), & \text{in } \Omega, \\ u'(x,0)=u_1(x), & \text{in } \Omega, \\ u'(x,t-\tau)=f_0(x,t-\tau), & \text{in } \Omega\times(0,\tau), \\ u=0, & \text{on } \partial\Omega\times\mathbb{R}^+. \end{cases} \tag{1.3}$$

Here Ω is a bounded domain in $\mathcal{R}^3$ with smooth boundary $\partial\Omega$ and (u_0,u_1,f_0) are given initial data. Let

$$h(s)=\begin{pmatrix} h_1(s) & 0 & 0 \\ 0 & h_2(s) & 0 \\ 0 & 0 & h_3(s) \end{pmatrix} \tag{1.4}$$

where $h_i:\mathbb{R}^+\to\mathbb{R}^+$ are given functions which represent the dissipative terms.

The qualitative studies of viscoelastic wave equations/systems have been many studied by many mathematicians and many results have been obtained in the last few years (see [1], [2], [3], [4], [5], [7], [9], [12]).

Without delay, a single viscoelastic wave equation was cosidered by [11] in the following Cauchy problem:

$$\begin{cases} u''-\Delta_x u+\int_0^t g(t-s)\Delta u(s,x)ds=0 & \text{in} \quad \mathbb{R}^n\times\mathbb{R}^+, \\ u(x,0)=u_0(x) \quad u'(x,0)=u_1(x) & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{1.5}$$

For a not necessarily decreasing relaxation function, the authors obtained a polynomial decay rate of solutions for compactly supported initial data. In [7], the authors studied the following equation :

$$u''-\Delta u+\int_0^t h(t-s)\Delta u(s,x)ds+b(x)u'+|u|^{p-1}u=0, \quad \Omega\times\mathbb{R}^+. \tag{1.6}$$

Here $b : \Omega \longrightarrow \mathbb{R}^+$ is a function, which may vanish on a part of the bounded domain Ω. By assuming $b(x) \geq b_0$ on $w \subset \Omega$ and for two positive constants ζ_1 and ζ_2 such that

$$-\zeta_1 h(t) \leq h'(t) \leq -\zeta_2 h(t) \tag{1.7}$$

under some geometry restrictions on w, the authors obtained an exponential decay result. In [5],the author established and extend the result in [6], under weaker conditions on both a and g, to a system where a source term is competing with the damping term. In order to compensate the lack of Poincare's inequality in $\mathbb{R}^n$ and for a wider class of relaxation functions, Zennir in [15] used weighted spaces to establish a very general decay rate of solutions for viscoelastic wave equations of Kirchhoff-type in

$$\rho(x)\left(|u'|^{q-2}u'\right)' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t h(t-s)\Delta_x u(s)ds = 0, x \in \mathbb{R}^n, t > 0 \tag{1.8}$$

where $q, n \geq 2$ and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$ and the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 is assumed to satisfy

$$m_0 - \overline{g} = l > 0, \quad g(0) = g_0 > 0 \tag{1.9}$$

where $\overline{g} = \int_0^\infty g(t)dt$. In addition, there exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$g'(t) + H(g(t)) \leq 0, t \geq 0, \quad H(0) = 0 \tag{1.10}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. Bchatnia and Daoulatli [1] considered the case of the Lamé system in a three-dimensional bounded domain with local nonlinear damping and external force, and obtained several boundedness and stability estimates depending on the growth of the damping and the external forces. The control region considered in [1] satisfies the famous geometric optical condition (GOC).

In Section 2, one of the main goal is to prove the global existence and uniqueness of solutions of (1.2)-(1.3). Section 3 is devoted to state and prove the main results of this work, that is, the stability of the system (1.2)-(1.3).

2. Well-posedness and uniqueness of solution

To prove the well-posedness and uniqueness of solutions of (1.2)-(1.3) using semi-group theory, we first consider the following hypothesis:

A1: The functions h_i is integrable on $\mathbb{R}^+$ and is such that

$$\mu - \int_0^{+\infty} h_i(s)ds > 0 \text{ and } \infty > \alpha_i = \int_0^{+\infty} h_i(s)ds > 0 \quad i = 1, 2, 3. \tag{2.11}$$

A2: The function h is of class $C^1(\mathbb{R}^+)$ and satisfies

$$h_i'(s) \leq \gamma_i h(s) \quad \forall s \in \mathbb{R}^+ \quad i = 1, 2, 3. \tag{2.12}$$

for a positive constants γ_i.

Following a methods developed in [8], [13], we consider two new auxiliary variables η and z, such that

$$\begin{cases} \eta(t,s) = u(t) - u(t-s) & \forall t, s \in \mathbb{R}^+ \\ \\ \eta_0(s) = \eta(0,s) = u(0) - u_0(s) & \forall s \in \mathbb{R}^+ \\ \\ z(t,\rho) = u_t(t - \tau\rho) & \forall t \in \mathbb{R}^+, \forall \rho \in (0,1) \\ \\ z_0(\rho) = z(0,\rho) = f_0(-\tau\rho) & \forall \rho \in]0,1[\end{cases} \tag{2.13}$$

Then, we have

$$\begin{cases} \eta_t(t,s) + \eta_s(t,s) = u_t(t) & \forall t, s \in \mathbb{R}^+ \\ \\ \eta(t,0) = 0 & \forall t \in \mathbb{R}^+ \end{cases} \tag{2.14}$$

and

$$\begin{cases} \tau z_t(t,\rho) + z_\rho(t,\rho) = 0 & \forall t \in \mathbb{R}^+, \forall \rho \in (0,1) \\ \\ z(t,0) = u'(t) & \forall t \in \mathbb{R}^+ \end{cases} \tag{2.15}$$

By combining (1.2) and (2.13), we obtain the following equation:

$$u'' - \left(\mu Id - \int_0^{+\infty} h(s)ds\right)\Delta u - (\lambda+\mu)\nabla \text{div } u - \int_0^{+\infty} h(s)\Delta\eta ds + kz(t,1) = 0 \text{ in } \Omega \times \mathbb{R}^+ \tag{2.16}$$

where

$$Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

Let the Hilbert spaces $L_h(\mathbb{R}^+, (H_0^1(\Omega))^3)$ defined by

$$L_h(\mathbb{R}^+, (H_0^1(\Omega))^3)) = \left\{ v = (v_1, v_2, v_3)^T : \mathbb{R}^+ \to (H_0^1(\Omega))^3, \int_0^{+\infty} h_i(s)\|\nabla v_i(s)\|^2 ds < +\infty \right\}$$

supplemented with the inner product

$$\langle v, w\rangle_{L_h} = \sum_{i=1}^{3} \int_0^{+\infty} h_i(s) \int_\Omega \nabla v_i(s).\nabla w_i(s) dx ds$$

for some $w = (w_1, w_2, w_3)^T$ and

$$L^2(]0,1[, L^2(\Omega)) = \left\{ v :]0,1[\to L^2(\Omega), \int_0^1 \|v(\rho)\|^2 d\rho < +\infty \right\},$$

endowed with the inner product

$$\langle v, w\rangle_{L^2(]0,1[,L^2(\Omega))} = \int_0^1 \int_\Omega v(\rho).w(\rho) dx d\rho.$$

Next, we will rewrite the system (1.2)-(1.3) in the following related system:

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t) \quad \forall t > 0, \\ \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{2.18}$$

where $\mathcal{U} = (u, u_t, \eta, z)^T$, $\mathcal{U}_0 = (u_0, u_1, \eta_0, z_0)^T \in \mathcal{H}$

$$\mathcal{H} = H_0^1(\Omega) \times (L^2(\Omega)) \times L_g(\mathbb{R}^+, H_0^1(\Omega)) \times L^2(]0,1[, L^2(\Omega)).$$

The operator $\mathcal{A}$ is linear and given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Delta_e - \left(\int_0^{+\infty} h(s)ds\right)\Delta & 0 & \int_0^{+\infty} h(s)\Delta ds & -\mu \\ 0 & 1 & -\partial_s & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau}\partial_\rho \end{pmatrix}. \tag{2.19}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ W = (w_1, w_2, w_3, w_4)^T \in \mathcal{H}, \mathcal{A}W \in \mathcal{H}, w_3(0) = 0 \text{ and } w_4(0) = w_2 \right\}.$$

The well-posedness and uniqueness of the problem (2.18) is given in.

Theorem 2.1. *Let the assumption (A1) and (A2) hold. Then, the system* (2.18) *has a unique weak solution for any* $\mathcal{U}_0 \in \mathcal{H}$, *such that*

$$\mathcal{U} \in C(\mathbb{R}^+, \mathcal{H}).$$

If $\mathcal{U} \in D(\mathcal{A})$, then the solution of (2.18) *satisfies (classical solution)*

$$\mathcal{U} \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C(\mathbb{R}^+, D(\mathcal{A})).$$

Proof. We prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator; that is, $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. Indeed, a simple calculation implies that, for any $V = (v_1, v_2, v_3, v_4)^T \in D(\mathcal{A})$,

$$\begin{aligned} \langle \mathcal{A}V, V\rangle_{\mathcal{H}} &= \sum_{i=1}^{3} \left\langle \Delta_e v_1^i - \int_0^{+\infty} h_i(s)\Delta v_3^i ds - k v_4^i(1), v_2^i \right\rangle + (\lambda+\mu)\langle \operatorname{div} v_2, \operatorname{div} v_1\rangle \\ &\quad + \sum_{i=1}^{3} (\mu - \alpha_i)\langle \nabla v_2^i, \nabla v_1^i\rangle + \left\langle -\frac{\partial v_3}{\partial s} + v_2, v_3 \right\rangle_{L_g^2} + \tau|\mu| \left\langle -\frac{1}{\tau}\frac{\partial v_4}{\partial \rho}, v_4 \right\rangle_{L^2(]0,1[,H)} \\ &\leq \frac{1}{2}\sum_{i=1}^{3} \int_0^{+\infty} h_i'(s) \int_\Omega |\nabla v_3^i|^2 dx ds \leq 0. \end{aligned} \tag{2.20}$$

since g_i nonincreasing. This implies that $\mathcal{A}$ is dissipative. On the other hand, we prove that $Id - \mathcal{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ we show that there exists $V = (v_1, v_2, v_3, v_4)^T \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})V = F. \tag{2.21}$$

This is equivalent to

$$\begin{cases} v_2 = v_1 - f_1, \\ v_3 + \frac{\partial v_3}{\partial s} = f_3 + v_1 - f_1, \\ v_4 + \frac{1}{\tau}\frac{\partial v_4}{\partial \rho} = f_4, \\ (\Delta_e + (1+|k|)Id)v_1 + \int_0^{+\infty} h(s)\Delta v_3 ds = (1+|k|)f_1 + f_2 - k v_4(1). \end{cases} \tag{2.22}$$

Noting that the second equation in $(2.22)_2$ with $v_3(0) = 0$ admits a unique solution

$$v_3 = (_0^s e^y (f_3(y) + v_1 - f_1) dy)\, e^{-s} \tag{2.23}$$

Eq.$(2.22)_3$ with $v_4(0) = v_2 = v_1 - f_1$ has a unique solution

$$v_4 = \left(v_1 - f_1 + \tau \int_0^\rho f_4(y) e^{\tau y} dy \right) e^{-\tau\rho}. \tag{2.24}$$

By (2.23) and (2.24) the Eq$(2.22)_4$ becomes

$$(l\Delta_e + (1+|k|) + e^{-\tau}k)Id)v_1 = \widetilde{f}, \tag{2.25}$$

where

$$l = \int_0^{+\infty} h(s)e^{-s}\left(\int_0^s e^y dy\right) ds = 1 - \int_0^{+\infty} h(s)e^{-s}ds$$

and

$$\widetilde{f} = f_2 + (1+|k|+e^{-\tau}k)f_1 - \int_0^s g(s)e^{-s}\left(\int_0^s e^y\Delta(f_3(y)-f_1)dy\right) ds - \tau k e^{-\tau}\int_0^1 f_4(y)e^{\tau y}dy.$$

We have just to prove that (2.25) has a solution $w_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ and replace in (2.23), (2.24) and the first equation in (2.22) to obtain $V \in D(\mathcal{A})$ satisfying (2.21). So we multiply (2.25) by a test function $\varphi_1 \in (H_0^1(\Omega))^3$ and we integrate by parts, obtaining the following variational formulation of (2.25):

$$a(v_1,\varphi_1) = L(\varphi_1) \quad \forall \varphi_1 \in (H_0^1(\Omega))^3, \tag{2.26}$$

where

$$\begin{aligned} a(v_1,\varphi_1) = \quad & \textstyle\int_\Omega \left(v_1.\varphi_1 + \sum_{i=1}^3(\mu-\alpha_i)\nabla v_1.\nabla\varphi_1 + (\lambda+\mu)\mathrm{div}\ v_1.\mathrm{div}\ \varphi_1\right) dx \\ & + \textstyle\int_\Omega (1+|k|) + e^{-\tau}k)Id)v_1.\varphi_1 dx \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} L(\varphi_1) = \quad & \textstyle\int_\Omega \left(f_2 + (1+|k|+e^{-\tau}k)f_1 - \tau k e^{-\tau}\int_0^1 f_4(y)e^{\tau y}dy\right)\varphi_1 dx \\ & + \textstyle\int_\Omega \int_0^s h(s)e^{-s}\left(\int_0^s e^y\nabla(f_3(y)-f_1)dy\right) ds.\nabla\varphi_1 dx. \end{aligned} \tag{2.28}$$

It is clear that a is a bilinear and continuous form on $(H_0^1(\Omega))^3 \times (H_0^1(\Omega))^3$, and L is a linear and continuous form on $(H_0^1(\Omega))^3$. On the other hand, (1.1) and (2.11) imply that there exists a positive constant a_0 such that

$$a(v_1,v_1) \geq a_0\|v_1\|_{(H_0^1(\Omega))^3}, \quad \forall v_1 \in (H_0^1(\Omega))^3,$$

which implies that a is coercive. Therefore, using the Lax-Milgram Theorem, we conclude that (2.26) has a unique solution $v_1 \in (H_0^1(\Omega))^3$. We then conclude that the solution v_1 of (2.26) belongs into $(H^2(\Omega)\cap H_0^1(\Omega))^3$ and satisfies (2.25). Consequently, using (2.23) and (2.24), we deduce that (2.21) has a unique solution $V \in D(\mathcal{A})$, this ensured that $Id - \mathcal{A}$ is surjective. Eqs.(2.20) and (2.21) inform us that $-\mathcal{A}$ is maximal monotone operator. By Lummer-Phillips theorem (see [14]), we deduce that $\mathcal{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$.

3. Stability results

We now define the classical energy of any weak solution u of (1.2)-(1.3) at time t as

$$E_u(t) = \frac{1}{2}\int_\Omega \left(\sum_{i=1}^{3}(\mu-\alpha_i)|\nabla u_i|^2 + (\lambda+\mu)|\text{div } u|^2 + |u'|^2\right)dx + \frac{\tau|k|}{2}\int_\Omega\int_0^1 z^2(t,\rho)d\rho dx$$

The *modifed* energy functional of the weak solution u is defined by

$$E(t) = E_u(t) + \frac{1}{2}h\circ\nabla\eta, \quad \forall t\in\mathbb{R}^+ \tag{3.29}$$

where

$$h\circ\nabla\eta = \sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega |\nabla u_i(x,t) - u_i(x,t-s)|^2 dxds \tag{3.30}$$

The next theorem is our main stability result.

Theorem 3.1. *Assume that (A1), (A2) and* (1.1) *hold. Then there exists a positive constant δ_0 independent of k such that, if*

$$|k| < \delta_0,$$

then, for any $\mathcal{U}_0\in\mathcal{H}$, there exist a positive constants δ_1 and δ_2, such that the solution of (2.18) *satisfies*

$$E(t)\le \delta_2 e^{-\delta_1 t} \quad \forall t\in\mathbb{R}^+. \tag{3.31}$$

The proof of Theorem 3.1 based on several Lemmas. The next Lemma means that our problem is dissipatif.

Lemma 3.1. *The functional (3.29) satisfies, along the solution u of* (1.2)-(1.3),

$$E'(t)\le \frac{1}{2}h'\circ\nabla\eta + |k|\int_\Omega |u'(t)|^2dx, \quad \forall t\in\mathbb{R}^+. \tag{3.32}$$

Proof. The multiplication of (1.2) by u', integrating by parts over Ω, we get easily (3.32).

In order to introduce an appropriate Lyapunov functional, we introduce the estimates.

Lemma 3.2. *The functional*

$$\phi(t) = \int_\Omega uu'dx, \quad \forall t\in\mathbb{R}^+ \tag{3.33}$$

satisfies for any $\varepsilon>0$

$$\begin{aligned}\phi'(t)\le\ & \textstyle\int_\Omega |u'|^2dx - \sum_{i=1}^3(\mu-\varepsilon-\alpha_i)\int_\Omega|\nabla u_i|^2dx \\ & -(\lambda+\mu)\textstyle\int_\Omega |div\, u|^2dx - k\int_\Omega z(t,1).udx + \frac{c_1}{4\varepsilon}h\circ\nabla\eta.\end{aligned} \tag{3.34}$$

Proof. By differentiating (3.33) and using (2.16), and (3.30), we obtain

$$\begin{aligned}\phi'(t) = \; & \int_\Omega |u'|^2 dx - \sum_{i=1}^3 (\mu - \alpha_i) \int_\Omega |\nabla u_i|^2 dx - (\lambda+\mu) \int_\Omega |\text{div } u|^2 dx \\ & + |k| \int_\Omega z(t,1).u dx - \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega \nabla u_i . \nabla \eta_i dx ds.\end{aligned} \tag{3.35}$$

By using Cauchy-Schwarz and Young's inequalities for the last term of (3.35), we obtain

$$- \int_0^{+\infty} h(s) \int_\Omega \nabla u . \nabla \eta dx ds \leq \; \varepsilon \int_\Omega |\nabla u|^2 dx + \tfrac{c_1}{4\varepsilon} h \circ \nabla \eta. \tag{3.36}$$

Inserting the last inequalities in (3.35), we obtain (3.34).

Lemma 3.3. *The functional*

$$\psi(t) = - \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i' . \eta_i dx ds, \quad \forall t \in \mathbb{R}^+. \tag{3.37}$$

satisfies for any $\varepsilon > 0$

$$\begin{aligned}\psi'(t) \leq \; & - \sum_{i=1}^3 (\alpha_i - \varepsilon) \int_\Omega |u_i'|^2 dx + \varepsilon \int_\Omega \left(|\nabla u|^2 + |div\, u|^2 \right) dx \\ & + \tfrac{c_1}{\varepsilon} h \circ \nabla \eta - \tfrac{c_2}{\varepsilon} h' \circ \nabla \eta + k \int_\Omega z(t,1). \left(\int_0^{+\infty} h(s) \eta ds \right) dx.\end{aligned} \tag{3.38}$$

Proof. Multiplying (2.16) by $\int_0^{+\infty} h(s)\eta(t,s) ds$ and integrating over Ω, we get

$$\begin{aligned}0 = \; & \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i'' . \eta_i dx ds \\ & - \sum_{i=1}^3 \int_\Omega (\mu \Delta u_i + (\lambda+\mu) \nabla \text{div } u_i) \left(\int_0^{+\infty} h_i(s) \eta_i ds \right) dx \\ & + \sum_{i=1}^3 \int_\Omega \left(\int_0^{+\infty} h_i(s) \Delta \eta_i ds \right) \left(\int_0^{+\infty} h_i(s) \eta_i ds \right) dx.\end{aligned} \tag{3.39}$$

By using the fact that, $\partial_t \eta(t,s) = -\partial_s \eta(t,s) + u'(t)$, we find

$$\begin{aligned} & \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i'' \eta_i dx ds \\ & = \sum_{i=1}^3 \left(\frac{\partial}{\partial t} \int_0^{+\infty} h_i(s) \int_\Omega u_i' \eta_i dx ds - \int_0^{+\infty} h_i(s) \int_\Omega u_i \eta_i' dx ds \right) \\ & = -\psi'(t) - \sum_{i=1}^3 \alpha_i \int_\Omega |u_i|^2 dx + \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i \partial_s \eta dx ds.\end{aligned} \tag{3.40}$$

By the fact that $\lim\limits_{s \to +\infty} h_i(s) = 0$ and $\eta_i(t,0) = 0$, integration with respect to s, we obtain

$$\sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i'' \eta_i dx ds = -\psi'(t) - \sum_{i=1}^3 \alpha_i \int_\Omega |u_i'|^2 + \sum_{i=1}^3 \int_\Omega u_i' \left(\int_0^{+\infty} h_i(s) \partial_s \eta_i ds \right) dx \tag{3.41}$$

Exploiting (3.39) and (3.41), we deduce that

$$\begin{array}{rl}\psi'(t)= & -\sum_{i=1}^{3}\alpha_i\int_\Omega|u_i|^2+k\sum_{i=1}^{3}\int_\Omega z(t,1).\left(\int_0^{+\infty}h_i(s)\eta_i ds\right)dx\\ & -\sum_{i=1}^{3}\int_\Omega u_i'\left(\int_0^{+\infty}h_i'(s)\eta_i ds\right)dx\\ & +\sum_{i=1}^{3}\int_\Omega((\mu-\alpha_i)\nabla u_i+(\lambda+\mu)\mathrm{div}\ u_i)\int_0^{+\infty}h_i(s)\nabla\eta_i dsdx\\ & +\sum_{i=1}^{3}\int_\Omega\left(\int_0^{+\infty}h_i(s)\nabla\eta_i ds\right).\left(\int_0^{+\infty}h_i(s)\nabla\eta_i ds\right)dx.\end{array} \tag{3.42}$$

Thanks to Cauchy-Schwarz and Young's inequalities to get

$$-\sum_{i=1}^{3}\int_\Omega u_i'\left(\int_0^{+\infty}h_i'(s)\eta_i ds\right)dx\le\varepsilon\sum_{i=1}^{3}\int_\Omega|u_i'|^2dx-\sum_{i=1}^{3}\frac{h_i(0)}{4\varepsilon}h_i'\circ\nabla\eta_i. \tag{3.43}$$

$$\sum_{i=1}^{3}\int_\Omega\nabla u_i\int_0^{+\infty}h_i(s)\nabla\eta_i dsdx\le\varepsilon\sum_{i=1}^{3}\int_\Omega|\nabla u_i|^2dx+\sum_{i=1}^{3}\frac{\alpha_i(1-\alpha_i)^2}{4\varepsilon}h_i\circ\nabla\eta_i$$

and

$$\sum_{i=1}^{3}\int_\Omega\left(\int_0^{+\infty}h_i(s)\nabla\eta_i ds\right)^2dx\le\alpha_i h_i\circ\nabla\eta_i.$$

Then, the proof is completes.

Lemma 3.4. *Let us define functional*

$$I(t)=\int_0^L\int_\Omega e^{-2\tau\rho}z(t,\rho)d\rho dx,\quad\forall t\in\mathbb{R}^+ \tag{3.44}$$

satisfy

$$I'(t)\le-2e^{-2\tau}\int_\Omega\int_0^1|z(t,\rho)|^2d\rho dx+\frac{1}{\tau}\int_\Omega|u'|^2dx-\frac{e^{-2\tau}}{\tau}\int_\Omega|z(t,1)|^2dx. \tag{3.45}$$

Proof. By Eq.(2.15), the derivative of I gives

$$\begin{array}{rl}I'(t) & =2\int_0^1e^{-2\tau\rho}\langle z'(t,\rho),z(t,\rho)\rangle d\rho\\ & =-\frac{2}{\tau}\int_0^1e^{-2\tau\rho}\langle\partial_\rho z(t,\rho),z(t,\rho)\rangle d\rho\\ & =-\frac{1}{\tau}\int_0^1e^{-2\tau\rho}\frac{\partial}{\partial_\rho}\|z(t,\rho)\|^2d\rho.\end{array} \tag{3.46}$$

Then by the fact that $z(t,0)=u'(t)$, we have

$$I'(t)=-2\int_\Omega\int_0^1e^{-2\tau\rho}|z(t,\rho)|^2d\rho dx+\frac{1}{\tau}\int_\Omega|u'|^2dx-\frac{e^{-2\tau}}{\tau}\int_\Omega|z(t,1)|^2dx,$$

which leads our result, since $-e^{-2\tau_1\rho}\le-e^{-2\tau_1}$, for any $\rho\in(0,1)$.

Now, we are ready to prove our main stability results (3.31).

Proof Let $L(t) = N_1E(t) + N_2\phi(t) + \psi(t) + I(t)$, for $N_1, N_2 > 0$. By definition of φ, ψ, I and E, there exist two constants d_1 and d_2 such that

$$d_1E(t) \leq L(t) \leq d_2E(t). \tag{3.47}$$

On the other hand, combining (3.32), (3.33), (3.37) and (3.45), we obtain

$$\begin{aligned} L'(t) \leq \; & \left(\tfrac{N_1}{2} - \tfrac{c_1}{\varepsilon}\right) h' \circ \nabla\eta + \left(\tfrac{N_2 c}{4\varepsilon} + \tfrac{c_1}{\varepsilon}\right) h \circ \nabla\eta - \textstyle\sum_{i=1}^{3}(\alpha_i - \varepsilon - N_2 - \tfrac{1}{\tau}) \int_\Omega |u_i'|^2 dx \\ & -(\lambda+\mu)\textstyle\int_\Omega |\text{div } u|^2 dx - \sum_{i=1}^{3}\left(N_2(\mu - \varepsilon - \alpha_i) - (1+\widehat{c})\varepsilon\right)\int_\Omega |\nabla u_i|^2 dx \\ & -2e^{-2\tau}\textstyle\int_\Omega\int_0^1 |z(t,\rho)|^2 d\rho dx - \frac{e^{-2\tau}}{\tau}\int_\Omega |z(t,1)|^2 dx \\ & +k\left\langle z(t,1), -N_2u(t) + \textstyle\int_0^{+\infty} h(s)\eta ds\right\rangle \end{aligned} \tag{3.48}$$

where $\widehat{c} > 0$ satisfies

$$\int_\Omega |\text{div } u|^2 dx \leq \widehat{c}\int_\Omega |\nabla u|^2 dx.$$

Next, the use of Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned} & k\left\langle z(t,1), -N_2u(t) + \textstyle\int_0^{+\infty} h(s)\eta ds\right\rangle \\ & \leq \varepsilon_1\|z(t,1)\|^2 + \tfrac{k^2}{4\varepsilon_1}\left(N_2\|u(t)\| + \textstyle\int_0^{+\infty} h(s)\|\eta(t,s)\|ds\right)^2 \\ & \leq \varepsilon_1\|z(t,1)\|^2 + \tfrac{k^2 c}{4\varepsilon_1}\left(N_2^2\|\nabla u(t)\|^2 + \alpha_i\textstyle\int_0^{+\infty} h(s)\|\nabla\eta(t,s)\|^2 ds\right) \\ & \leq \varepsilon_1\|z(t,1)\|^2 + \varepsilon_1 k^2 c_6\left(\|\nabla u(t)\|^2 + \alpha_i\textstyle\int_0^{+\infty} h(s)\|\nabla\eta(t,s)\|^2 ds\right), \end{aligned} \tag{3.49}$$

where $c_6 = \max\{N_2^2, \alpha_i\}$. We now choose $0 < \varepsilon < \mu - \max_{1\leq i\leq 3}\{\alpha_i\}$ and $0 < \varepsilon_1 < \min_{1\leq i\leq 3}\{\alpha_i\}$. Next, we choose N_2 and ε_2 such that $0 < N_2 < \min_{1\leq i\leq 3}\{\alpha_i\} - \varepsilon_1 - \frac{1}{\tau}$ and $0 < \varepsilon_2 < \frac{N_2}{1+\widehat{c}}(\mu - \max_{1\leq i\leq 3}\{\alpha_i\} - \varepsilon)$. These choices imply that $\alpha_i - \varepsilon_1 - N_2$ and $(N_2(\mu - \varepsilon - \alpha_i) - (1+\widehat{c})\varepsilon_2)$ are positive constants. Then, we obtain, for some $\beta, c_3, c_4 > 0$,

$$L'(t) \leq -\beta E(t) + \left(\frac{N_1}{2} - c_3\right) h' \circ \nabla\eta + c_4 h \circ \nabla\eta, \quad \forall t \in \mathbb{R}^+. \tag{3.50}$$

Finally, we can choose N_1 large enough so that $\frac{N_1}{2} - c_3 \geq 0$

$$L'(t) \leq -\beta E(t) + c_4 h \circ \nabla\eta, \quad \forall t \in \mathbb{R}^+. \tag{3.51}$$

If (2.12) is satisfied, for any $i = 1, 2, 3$, then

$$h_i \circ \nabla\eta_i \leq -\frac{1}{\delta_i} h_i' \circ \nabla\eta_i. \tag{3.52}$$

Combing (3.53) and (3.52) imply that

$$L'(t) \leq -\beta E(t) - c_5 h' \circ \nabla\eta, \quad \forall t \in \mathbb{R}^+. \tag{3.53}$$

where $c_5 = c_4 \max\{\delta_i\}$

Let $F = L + c_5 E$. Using (3.32), we get

$$F'(t) \leq -\beta E(t) \quad \forall t \in \mathbb{R}^+. \tag{3.54}$$

Because $L \sim E$, then $F \sim E$. Therefore, (3.54) implies that

$$F'(t) \leq -c' F(t) \quad \forall t \in \mathbb{R}^+$$

for some $c' > 0$. By integrating this differential inequality, we get

$$F(t) \leq F(0) e^{-c't}, \quad \forall t \in \mathbb{R}^+$$

Thus, thanks to $F \sim E$, we get (3.31).

References

[1] A. Bchatnia and M. Daoulatli, *Behavior of the energy for Lamé systems in bounded domains with nonlinear damping and external force*, Electron. J. Dif. Equa., (2013), 17.

[2] A. Bchatnia and A. Guesmia *well-posedness and asymptotic stability for the lamé system with infinite memories in a bounded domain* Math. Cont. And Related Fields Vol. 4, 2014.

[3] Benaissa, A. and Aissa Benguessoum *Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the non-linear internal feedback* Int. J. Dynamical Systems and Differential Equations, Vol. 5, No. 1, 2014

[4] A. Beniani, A. Benaissa and Kh. Zennir, *Polynomial decay of solutions to the Cauchy problem for a Petrowsky-Petrowsky system in* $\mathbb{R}^n$, Acta. Appl. Math., Vo 146, (1) (2016), 67-79.

[5] Berrimi, S., Messaoudi, S.A., *Existence and decay of solutions of a viscoelastic equation with a nonlinear source.* Nonl. Anal. 64, (2006), 2314-2331.

[6] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: *Exponential decay for the solution of semi-linear viscoelastic wave equations with localized damping.* E. J. Differ. Equ. 44, (2002), 1-14.

[7] Cavalcanti, M.M, Domingos Cavalcanti, V.N, Ferreira, J: *Existence and uniform decay for a non-linear viscoelastic equation with strong damping.* Math. Methods Appl. Sci. 24, (2001), 1043-1053.

[8] Dafermos, C.M. *Asymptotic stability in viscoelasticity.* Arch. Ration. Mech. Anal. 37, (1970), 297-308.

[9] Fridman, E., Nicaise, S. and Valein, J. *Stabilization of second order evolution equations with unbounded feedback with time-dependent delay.* SIAM J. Control Optim., 8, (2010), 5028-5052.

[10] A. Guesmia, *Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay* Journal of Mathematical Control and Information, (2013), 1-20, doi:10.1093/imamci/dns039.

[11] Kafini, M, Tatar, N-e: *A decay result to a viscoelastic in Rn with an oscillating kernel.* J. Math. Phys. 51(7), 073506 (2010)

[12] A. Guesmia and S. Messaoudi, *A general decay result for a viscoelastic equation in the presence of past and infinite history memories.* Nonlinear Analysis, 13 (2012), 476-485.

[13] Nicaise, S., Pignotti, C. *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks.* SIAM J. Control Optim. 45, (2006), 1561-1585

[14] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* New York: Springer, (1983).

[15] Zennir, Kh. *General decay of solutions for damped wave equation of Kirchhoff type with density in* $\mathbb{R}^n$.Ann Univ Ferrara, 61, (2015) 381-394.

[16] Zennir, Kh., Bayoud M. and Svetlin G. G. *Decay of Solution for DegenerateWave Equation of Kirchhoff Type in Viscoelasticity.*Int. J. Appl. Comput. Math (2018) 4:54.

[17] S. Zitouni and Kh. Zennir, *On the existence and decay of solution for viscoelastic wave equation with nonlinear source in weighted spaces.* Rend. Circ. Mat. Palermo, II. Ser, (2017) 66, pp 337-353.

A. Beniani. Laboratory ACEDP, University of Djillali Liabes - B. P. 89, Sidi Bel Abbes 22000, Algeria

Email address: a.beniani@yahoo.fr

Kh. Zennir. First: Laboratory LAMAHIS, University 20 Aout 1955, Skikda 21000, ALGERIA.

Second: Department Of Mathematics, College Of Sciences and Arts, Al-Ras, Al-Qassim University, Kingdom Of Saudi Arabia.

Email address: k.zennir@qu.edu.sa

A. Benaissa. Laboratory ACEDP, University of Djillali Liabes - B. P. 89, Sidi Bel Abbes 22000, Algeria

Email address: benaissa_abbes@yahoo.com

S. Boulaares. Department Of Mathematics, College Of Sciences and Arts, Al-Ras, Al-Qassim University, Kingdom Of Saudi Arabia

Email address: salehboulaares@yahoo.com

International Journal of Maps in Mathematics
Volume:1, Issue:2, (2018), Pages:188-218
ISSN: 2636-7467 (Online)
www.journalmim.com

TIMELIKE SYMMETRIES AND CAUSALITY IN LORENTZIAN MANIFOLDS

C. ATINDOGBÉ* AND R. HOUNNONKPE

Abstract. Metric and curvature symmetries (Killing, homothetic, conformal, null convergence conditions...) of Riemannian, semi-Riemannian, and lightlike manifolds play an important role in theoretical physics, especially in general relativity. In the present paper, we investigate and discuss the consequences that spacetimes admit such symmetries and show that their existence places restrictions on both the null geometry of hypersurfaces and the different hierarchies of spacetime causality.

1. Introduction

Metric symmetries and curvature conditions (such as null convergence conditions) of Riemannian, semi-Riemannian, and lightlike manifolds play an important role in theoretical physics, especially in general relativity ([9, 12, 18, 16, 19] and references therein). The purpose of this paper is to focus on the consequences of the existence of some symmetries (timelike conformal, homothetic, Killing, affine Killing, affine conformal Killing, projective vector fields) on both the geometry of null hypersurfaces and causal hierarchy of spacetimes (chronology, total imprisoning, stably causal, causaly continuous, strongly causal, totally vicious, reflecting...).

Received:2018-05-14 *Revised:2018-07-17* *Accepted:2018-09-05*

*2010 Mathematics Subject Classification:*53C30, 83C75, 53C40.

Key words: Causality, timelike symmetries, null hypersurface, causal hierarchy

* Corresponding author

As we shall see, the existence of certain symmetries places important restrictions on the properties of null hypersurfaces (See Theorem 3.1, Theorem 3.2, Theorem 3.3 in Section 3) and the spacetime (for example Theorem 4.8 tells us among other that if a compact conformally flat Lorentzian manifold of dimension 4 with nowhere vanishing scalar curvature obeys the null convergence condition and supports a timelike affine conformal Killing vector field then it is totally vicious). The following organization is adopted for the paper. In Section 2 we summarize some elements of causality theory and the causal hierarchy of spacetimes and provide typical ingredients of the geometry of null hypersurfaces and rigged Riemannian structure. Most of these introductory materials are to be found in [20, 21, 22, 23, 8, 12]. In Section 3, after some technical results on the function $\tau(\xi)$ which represents the obstruction to the geodesibility of the rigged vector field, we prove a non-existence result of closed (in the topological sense) embedded null hypersurface (Theorem 3.3) and geodesibility properties (Theorem 3.1 and Theorem 3.2) on orientable Lorentzian manifolds admitting timelike affine conformal Killing vector field (resp. timelike projective vector field). In Section 4, we discuss causality in conformally flat spacetimes. The main results of this section are restrictions placed on some levels of the causal ladder of spacetimes such as mentioned above and located at Theorems 4.1, 4.2, 4.3, 4.4, 4.6, 4.7, 4.8 and Corollary 4.1. In Section 5, causality conditions are also explored restricted to quasi-Einstein spacetimes with some applications to perfect fluid spacetimes (Theorem 5.1,Theorem 5.2, Theorem 5.3,Theorem 5.5 and Theorem 5.4). In Section 6, we extend to Hubble-isotropic spacetimes (Theorem 6.2), under a non negative (resp. a non positive) assumption on the expansion, the following result [14]: conformally stationary spacetime with a complete stationary vector field is reflecting. Similar sufficient conditions based on the sign of the expansion are given in Theorem 6.4 and Theorem 6.7 to ensure that Hubble-isotropic spacetimes are distinguishing or stably causal.

2. Preliminaries

2.1. **Elements of Causality theory and the causal hierarchy.**

2.1.1. *Causality relations.* The causality relations on $\overline{M}$ are defined as follows. If $p, q \in \overline{M}$, then $p \ll q$ means there is a future-pointing timelike curve in $\overline{M}$ from P to q; $p < q$ means there is a future-pointing causal curve in $\overline{M}$ from p to q. Evidently $p \ll q$ implies $p < q$. As usual, $p \leq q$ means that either $p < q$ or $p = q$. For a subset A of $\overline{M}$, the subset $I^+(A) = \{q \in \overline{M} :$ there is a $p \in A$ with $p \ll q\}$ is called the chronological future of A, and $J^+(A) = \{q \in \overline{M} :$ there is a $p \in A$ with $p \leq q$ $\}$ is called the causal future of A. Thus

$A \cup I^+(A) \subset J^+(A)$. For a single point, $I^+(p) = \{q : p \ll q\}$; similarly for J^+. Dual to the preceding definitions are corresponding past versions. Thus $I^-(A) = \{q \in \overline{M} :$ there is a $p \in A$ with $q \ll p$ $\}$ is the chronological past of A. In general, past definitions and proofs follow from the future versions (and vice versa) merely by reversing time-orientation.

Definition 2.1. *A point $p \in \overline{M}$ is a future endpoint of a future-directed causal curve $\gamma : I \longrightarrow \overline{M}$ if, for every neighborhood $\mathcal{O}$ of p, there exists a point $t_0 \in I$ such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$. A causal curve is future inextensible (respectively, past inextensible) if it has no future (respectively, past) endpoint.*

Definition 2.2. *A future inextensible causal curve $\gamma : I \longrightarrow \overline{M}$, is totally future imprisoned in the compact set C if there is $t_0 \in I$, such that for every $t > t_0, t \in I, \gamma(t) \in C$, i.e. if it enters and remains in C. It is partially future imprisoned if for every $t_0 \in I$, there is $t > t_0, t \in I$, such that $\gamma(t) \in C$, i.e. if it continually returns to it. The curve escapes to infinity in the future if it is not partially future imprisoned in any compact set.*

2.1.2. *Causality conditions.* If $(\overline{M}, \overline{g})$ contains no closed timelike curves, we say that the chronology condition holds on $(\overline{M}, \overline{g})$. A spacetime $(\overline{M}, \overline{g})$ satisfies the causality condition provided there are no closed causal curves in $\overline{M}$. Obviously this implies the chronology condition, but not conversely. The causality condition (and similarly for chronology) is said to hold at a point p if there are no closed causal curves through p, and on a subset A if it holds at each $p \in A$. A spacetime is non-total future imprisoning if no future inextensible causal curve is totally future imprisoned in a compact set. A spacetime is non-partial future imprisoning if no future inextensible causal curve is partially future imprisoned in a compact set. Actually, Beem proved [5, Theorem 4] that a spacetime is non-total future imprisoning if and only if it is non-total past imprisoning, thus in the non-total case one can simply speak of the non-total imprisoning property (condition N, in Beem's terminology [5]). The strong causality condition holds at $p \in \overline{M}$ provided that given any neighborhood $\mathcal{U}$ of p there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that every causal curve segment with endpoints in $\mathcal{V}$ lies entirely in $\mathcal{U}$. $\overline{M}$ is strongly causal if the strong causality condition holds at each $p \in \overline{M}$. The following new step on the causal ladder has also been established.

Definition 2.3. *A spacetime $(\overline{M}, \overline{g})$ is called feebly distinguishing if $(p, q) \in J^+, p \in \overline{I^+(q)}$ and $q \in \overline{I^-(p)}$ implies $p = q$.*

A spacetime $(\overline{M}, \overline{g})$ is future-distinguishing at $p \in \overline{M}$ if and only if $I^+(p) \neq I^+(q)$ for each $q \in \overline{M}$, with $q \neq p$. $\overline{M}$ is future-distinguishing if and only if it is future-distinguishing at every point. This property of being future-distinguishing is called future-distinction. The concept of past-distinction is defined similarly. A spacetime is stably causal if it cannot be made to contain closed trips by arbitrarily small perturbations of the metric. The condition of stable causality is equivalent to the existence of a global time function on $(\overline{M}, \overline{g})$, that is to say, a function on $\overline{M}$ whose gradient is everywhere timelike and future-pointing. There is one condition, related in some ways to the causality conditions below, which stands, nevertheless, outside the causal ladder.

Definition 2.4. *A spacetime $(\overline{M}, \overline{g})$ is called reflecting if $I^+(q) \subset I^+(p) \Leftrightarrow I^-(p) \subset I^-(q)$ for all $p, q \in \overline{M}$.*

A spacetime $(\overline{M}, \overline{g})$ is called causally continuous if it is reflecting and feebly distinguishing. Usually (see [20]), causal continuity was defined as a spacetime being reflecting and distinguishing. In ([21]), it is proved that the assumption can be relaxed to feeble distinction. Causal continuity is stronger than stable causality. A spacetime $(\overline{M}, \overline{g})$ is called causally simple if it is causal and $J^+(p), J^-(p)$ are closed sets for all $p \in \overline{M}$. Finally, $(\overline{M}, \overline{g})$ is called globally hyperbolic if it is causal and $J^+(p) \cap J^-(p)$ are compact sets for all $p, q \in \overline{M}$.

2.2. **Geometry of null hypersurfaces and rigged Riemannian structure.** In this section, we review some facts about null hypersurfaces, see [8] for more details. Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in $\overline{M}$. A *screen distribution* on M^{n+1}, is a complementary bundle of $TM^{\perp}$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution. Each of them is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$. From [8], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0 \tag{2.1}$$

$\forall W \in \mathscr{S}(N)|_{\mathscr{U}}$.Then $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus \mathscr{S}(N). \tag{2.2}$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M. Now, we need to use the (general) concept of rigging for null hypersurfaces, see [12] for details.

Definition 2.5. *Let M be a null hypersurface in a Lorentzian manifold. A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.*

Given a rigging ζ in a neighborhood of M in $(\overline{M}, \overline{g})$ let α denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e $\alpha = \overline{g}(\zeta, \ .)$. Take $\omega = i^{\star}\alpha$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \alpha \otimes \alpha \quad \text{and} \quad \widetilde{g} = i^{\star}\breve{g}. \tag{2.3}$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on the (whole) hypersurface M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact the rigged vector field ξ is the unique lightlike vector field in M such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. A screen distribution on M is given by $\mathscr{S}(\zeta) = TM \cap \zeta^{\perp}$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field to $\mathscr{S}(\zeta)$ is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{2.4}$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M, ζ) (the latter is called a normalization of the null hypersurface). A normalization (M, ζ) is said to be closed (resp. conformal) if the rigging ζ is closed i.e the 1-form α is closed (resp. ζ is a conformal vector field, i.e there exists a function ρ on M such that $L_{\zeta}\overline{g} = 2\rho\overline{g}$). We say that ζ is a *null rigging* for M if the restriction of ζ to the null hypersurface M is a null vector field.

Let ζ be a rigging for a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$. The screen distribution $\mathscr{S}(\zeta) = \ker \omega$ is integrable whenever ω is closed, in particular if the rigging is closed. On a normalized null hypersurface (M, ζ), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.6}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -\overset{\star}{A}_{\xi} X - \tau(X)\xi, \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null transverse vector field N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(\zeta)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma\big(\mathscr{S}(\zeta)\big)$ with respect to the decomposition (2.7). Now the $(0,2)$ tensors B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM with respect to the rigging ζ and the rigged vector field ξ respectively and τ a 1-form on TM defined by

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi).$$

For the second fundamental forms B and C the following holds

$$B(X,Y) = g(\overset{\star}{A}_\xi X, Y), \quad C(X, PY) = g(A_N X, Y) \;\; \forall X, Y \in \Gamma(TM), \tag{2.9}$$

and

$$B(X,\xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0. \tag{2.10}$$

A null hypersurface M is said to be *totally umbilic* (resp. *totally geodesic*) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u, v \in T_pM$, $B(p)(u,v) = \rho(p)\overline{g}(u,v)$ (resp. B vanishes identically on M). These are intrinsic notions on any null hypersurface in the sense that they are independent of the normalization. Remark that M is *totally umbilic* (resp. *totally geodesic*) if and only if $\overset{\star}{A}_\xi = \rho P$ (resp. $\overset{\star}{A}_\xi = 0$). The trace of $\overset{\star}{A}_\xi$ is the lightlike (non normalized) mean curvature of M, explicitly given by

$$H_p = \sum_{i=2}^{n+1} \overline{g}(\overset{\star}{A}_\xi(e_i), e_i) = \sum_{i=2}^{n+1} B(e_i, e_i),$$

being $(e_2, \ldots, e_{n+1})$ an orthonormal basis of $\mathscr{S}(N)$ at p .

3. Timelike symmetries and rigging

3.1. **Timelike projective and affine conformal Killing vectors field.** In [12], several results using energy conditions and timelike conformal vector field have been proved. We extend this results to timelike affine conformal Killing vector field and timelike projective vector field.

Definition 3.1. 1. *A vector field ζ is called affine conformal Killing if $L_\zeta \overline{g} = \rho\overline{g} + K$ where K is a second order covariant constant ($\overline{\nabla}K = 0$) symmetric tensor field.*

2. *ζ is a projective vector field if*

$$(L_\zeta\overline{\nabla})(X,Y)=\mu(X)Y+\mu(Y)X, \forall\, X,Y\in T\overline{M}$$

where μ is a $1-$ forme defined on $\overline{M}$.

We prove the following.

Lemma 3.1. *Let $(\overline{M},\overline{g})$ be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field). For any null hypersurface M in $\overline{M}$, the normalized null hypersurface (M,ζ) satisfies*

$$\xi(\tau(\xi))+2(\tau(\xi))^2=0. \tag{3.11}$$

Proof. We consider first the case ζ is a timelike affine conformal Killing vector field. By definition, $L_\zeta\overline{g}=\rho\overline{g}+K$ where K is a second order covariant constant ($\overline{\nabla}K=0$) symmetric tensor field. From [12, Corollary 3.6], we have

$$\tau(\xi)=\overline{g}(\overline{\nabla}_\xi\zeta,\xi)=\frac{1}{2}(L_\zeta\overline{g})(\xi,\xi).$$

It follows that $\tau(\xi)=\frac{1}{2}K(\xi,\xi)$. Now as $\overline{\nabla}K=0$, we have $(\overline{\nabla}_\xi K)(\xi,\xi)=0$ which leads to $\xi(K(\xi,\xi)-2K(\xi,\overline{\nabla}_\xi\xi)=0$ and then $\xi(K(\xi,\xi)+2\tau(\xi)K(\xi,\xi)=0$. Finally since $\tau(\xi)=\frac{1}{2}K(\xi,\xi)$, we get $\xi(\tau(\xi))+2(\tau(\xi))^2=0$. Now, suppose ζ is a timelike projective vector field. By definition,

$$(L_\zeta\overline{\nabla})(X,Y)=\mu(X)Y+\mu(Y)X, \forall\, X,Y\in T\overline{M}$$

where μ is a $1-$ forme defined on $\overline{M}$. It follows that $(L_\zeta\overline{\nabla})(\xi,\xi)=2\mu(\xi)\xi$ that is

$$[\zeta,\overline{\nabla}_\xi\xi]-\overline{\nabla}_{[\zeta,\xi]}\xi-\overline{\nabla}_\xi[\zeta,\xi]=2\mu(\xi)\xi.$$

Since ξ is lightlike and $\overline{\nabla}_\xi\xi=-\tau(\xi)$ we get

$$-\tau(\xi)\overline{g}([\zeta,\xi],\xi)-\overline{g}(\overline{\nabla}_\xi[\zeta,\xi],\xi)=0$$

and

$$-\tau(\xi)\overline{g}([\zeta,\xi],\xi)-(\xi(\overline{g}([\zeta,\xi],\xi))+\tau(\xi)\overline{g}([\zeta,\xi],\xi))=0.$$

Taking into account that $\overline{g}([\zeta,\xi],\xi)=-\tau(\xi)$. We obtain

$$\xi(\tau(\xi))+2(\tau(\xi))^2=0.$$

From equation(3.11), it follows that the function $\tau(\xi)$ vanishes identically on M if the rigged vector field ξ is complete. This is the case when M is a compact null hypersurface. Hence we get the following corollary.

Corollary 3.1. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field). Let (M, ζ) be a normalized compact null hypersurface, then the rigged vector field ξ is $\overline{g}-$geodesic that is $\tau(\xi) = 0$.*

In [12] it is shown that if the reverse null convergence condition (that is $\overline{Ric}(U) \leq 0$ for all lightlike vector field U) holds on a Lorentzian manifold admitting a timelike conformal vector field then any compact totally umbilic null hypersurface is totally geodesic. Replacing the existence of a timelike conformal vector field by the existence of a timelike affine conformal Killing vector field (resp. a timelike projective vector field), we get the same result as stated in the following.

Theorem 3.1. *Let $(\overline{M}^{n+2}, \overline{g})$, with $n \geq 1$ be an orientable Lorentzian manifold such that $\overline{Ric}(U) \leq 0$ for all lightlike vector field U. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field (resp. a timelike projective vector field) ζ. Then any compact totally umbilic null hypersurface is totally geodesic.*

Proof. Let M be a compact totally umbilic null hypersurface with umbilicity factor ρ. Recall that

$$\overline{Ric}(\xi) = \xi(H) + \tau(\xi)H - |\overset{*}{A}_\xi|^2,$$

[3, Remark 3.] and $H = -\widetilde{div}(\xi)$. From Corollary 3.1, $\tau(\xi) = 0$. hence

$$\overline{Ric}(\xi) = \xi(H) - |\overset{*}{A}_\xi|^2. \tag{3.12}$$

It follows by integrating (3.12) that

$$\int_M \overline{Ric}(\xi) d\widetilde{g} = \int_M (\xi(H) - |\overset{*}{A}_\xi|^2) d\widetilde{g}.$$

By the divergence theorem,

$$\int_M (\xi(H) d\widetilde{g} = -\int_M H \widetilde{div}(\xi) d\widetilde{g} = \int_M H^2 d\widetilde{g}.$$

Hence

$$\int_M \overline{Ric}(\xi) d\widetilde{g} = \int_M (H^2 - |\overset{*}{A}_\xi|^2) d\widetilde{g} = \int_M n(n-1)\rho^2 d\widetilde{g} \geq 0.$$

Using $\overline{Ric}(\xi) \leq 0$, we get that $n(n-1)\rho^2$ vanishes identically. If $n \geq 2$ then ρ^2 vanishes identically and M is totally geodesic. If $n = 1$ then

$$\int_M \overline{Ric}(\xi) d\widetilde{g} = 0$$

and since $\overline{Ric}(\xi)$ has sign, $\overline{Ric}(\xi) = 0$. In this case (3.12) becomes

$$\xi(\rho) - \rho^2 = 0.$$

As ξ is complete (being M compact), $\rho = 0$ and the conclusion holds.

In case the null convergence condition holds we get the following.

Theorem 3.2. *Let $(\overline{M}^{n+2}, \overline{g})$, with $n \geq 1$ be a Lorentzian manifold satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field (resp. a timelike projective vector field) ζ . Then any compact null hypersurface in $\overline{M}$ is totally geodesic.*

Proof. Let M be a compact null hypersurface in $\overline{M}$. It holds

$$\overline{Ric}(\xi) = \xi(H) + \tau(\xi)H - |\overset{*}{A}_\xi|^2.$$

From Corollary 3.1, $\tau(\xi) = 0$, hence

$$\overline{Ric}(\xi) = \xi(H) - |\overset{*}{A}_\xi|^2.$$

The null convergence condition and the inequality $|\overset{*}{A}_\xi|^2 \geq \frac{1}{n}H^2$ lead to $\xi(H) - \frac{1}{n}H^2 \geq 0$, and since ξ is complete (M is compact) we get that $H = 0$. From the relation $\xi(H) - |\overset{*}{A}_\xi|^2 \geq 0$, it follows that $|\overset{*}{A}_\xi|^2 = 0$ which leads to $\overset{*}{A}_\xi = 0$. We conclude that M is totally geodesic.

More generally, we prove the following.

Theorem 3.3. *Let $(\overline{M}^{n+2}, \overline{g})$, with $n \geq 1$ be a null complete Lorentzian manifold such that $\overline{Ric}(U) > 0$ for all null vector $U \in T\overline{M}$. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field (resp. a timelike projective vector field) ζ. Then it can not exist any closed (in the topological sense) embedded null hypersurface.*

Proof. Suppose that M is a closed embedded null hypersurface in $(\overline{M}, \overline{g})$ and consider ζ as a rigging for M. From Lemma 3.1

$$\xi(\tau(\xi)) + 2(\tau(\xi))^2 = 0. \tag{3.13}$$

If $\tau(\xi)$ never vanishes on M then setting $\tilde{\xi} = \exp(\frac{1}{\sqrt{|\tau(\xi)|}})\xi$, it follows that $\tilde{\xi}$ is a geodesic null vector field tangent to M which is complete since $\overline{M}$ is null complete and M is closed embedded. To simplify notation, we still call $\tilde{\xi}$ by ξ. Then as ξ is geodesic, $\tau(\xi) = 0$ and

$$\overline{Ric}(\xi) = \xi(H) - |\overset{*}{A}_\xi|^2.$$

Since $\overline{Ric}(\xi) > 0$, the inequality $|\overset{*}{A}_\xi|^2 \geq \frac{1}{n}H^2$ lead to $\xi(H) - \frac{1}{n}H^2 > 0$, and since ξ is complete it follows that $H = 0$ which is a contradiction. Now, suppose $\tau(\xi)$ vanishes at some $p \in M$. Let γ_p be the integrale curve of ξ through p. γ_p is a complete geodesic curve and $(\tau(\xi) \circ \gamma)' + 2(\tau(\xi) \circ \gamma)^2 = 0$. As the unique solution of the differential equation $y' + 2y^2 = 0$ which can vanish is the trivial solution, we get $\tau(\xi) \circ \gamma = 0$. As above this leads to $(H \circ \gamma)' - \frac{1}{n}(H \circ \gamma)^2 > 0$, and since γ is complete it follows that $H \circ \gamma = 0$ which is a contradiction.

Before proving the next proposition, we need the following lemma.

Lemma 3.2. *Let (M, ζ) be a normalized null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$ such that ζ is affine conformal Killing. Then $\tau(X) = C(\xi, X) + K(\xi, X) \quad \forall\, X \in \mathscr{S}(\zeta)$. In particular if ζ is conformal then $\tau(X) = C(\xi, X), \quad \forall\, X \in \mathscr{S}(\zeta)$.*

Proof. Since ζ is affine conformal Killing there exists a function ρ on $\overline{M}$ such that $L_\zeta \overline{g} = \rho \overline{g} + K$. It follows that $(L_\zeta \overline{g})(\xi, X) = K(\xi, X) \quad \forall\, X \in \mathscr{S}(\zeta)$. Then $(L_\zeta \overline{g})(\xi, X) = \overline{g}(\overline{\nabla}_\xi \zeta, X) + \overline{g}(\overline{\nabla}_X \zeta, \xi) = K(\xi, X)$. Since $\overline{g}(\zeta, X) = 0, \overline{g}(\overline{\nabla}_\xi \zeta, X) = -\overline{g}(\overline{\nabla}_\xi X, \zeta)$. Using the fact that

$$\overline{\nabla}_\xi X = \nabla_\xi X = \overset{\star}{\nabla}_\xi X + C(\xi, X)\xi,$$

we get $\overline{g}(\overline{\nabla}_\xi X, \zeta) = C(\xi, X)$. Moreover, $\overline{g}(\overline{\nabla}_X \zeta, \xi) = \tau(X)$. It follows that $-C(\xi, X)\xi + \tau(X) = K(\xi, X)$ and then $\tau(X) = C(\xi, X) + K(\xi, X)$.

Proposition 3.1. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized compact null hypersurface such that ζ is an affine conformal Killing vector field satisfying $\overline{\nabla}_X(d\alpha) = 0, \quad \forall\, X \in \zeta^\perp$ where $\alpha = g(\zeta, .)$. If M is totally geodesic then it holds:*

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, X)X, N) \quad \forall\, X \in \mathscr{S}(\zeta). \tag{3.14}$$

Proof. From the Gauss-Codazzi equations, see [8, Page 95, Eq. (3.11)],

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X) \end{aligned}$$

$\forall\, X, Y, Z \in TM$ where

$$(\nabla_X C)(Y, PZ) = X(C(Y, PZ) - C(\nabla_X Y) - C(Y, \overset{\star}{\nabla}_X PZ).$$

So, we have

$$\overline{g}(\overline{R}(\xi, X)X, N) = (\nabla_\xi C)(X, X) - (\nabla_X C)(\xi, X) + C(\xi, X)\tau(X) - C(X, X)\tau(\xi) \qquad (3.15)$$

$\forall\, X \in \mathscr{S}(\zeta)$. From [12, Corollary 3.6 (4)] it holds

$$-2C(U, X) = d\alpha(U, X) + (L_\zeta \overline{g})(U, X) + \overline{g}(\zeta, \zeta)B(U, X)$$

$\forall\, U \in TM$ and $\forall\, X \in \mathscr{S}(\zeta)$. As M is totally geodesic,

$$-2C(U, X) = d\alpha(U, X) + (L_\zeta \overline{g})(U, X)$$

that is

$$C(U, X) = -\frac{1}{2}d\alpha(U, X) - \frac{1}{2}\rho\overline{g}(U, X) - \frac{1}{2}K(U, X). \qquad (3.16)$$

Being M compact and ζ affine conformal Killing, from Corollary 3.1, $K(\xi, \xi) = \tau(\xi) = 0$. Equation (3.15) can be written as

$$\begin{aligned}\overline{g}(\overline{R}(\xi, X)X, N) \quad &= \quad \xi(C(X, X)) - C(\nabla_\xi X, X) - C(X, \overset{\star}{\nabla}_\xi X) \\ &\quad\ -(X(C(\xi, X)) - C(\nabla_X \xi, X) - C(\xi, \overset{\star}{\nabla}_X X)) + C(\xi, X)\tau(X)\end{aligned}$$

$\forall X \in \mathscr{S}(\zeta)$. Using (3.16), $\overline{\nabla}K = 0$ and $\overline{\nabla}_X(d\alpha) = 0$, we get

$$\overline{g}(\overline{R}(\xi, X)X, N) = (\xi(-\frac{\rho}{2})\overline{g})(X, X) - \frac{1}{2}C(\xi, X)K(\xi, X) + \frac{1}{2}C(\xi, X)d\alpha(\xi, X) + C(\xi, X)\tau(X).$$

From (3.16) $C(\xi, X) = -\frac{1}{2}d\alpha(\xi, X) - \frac{1}{2}K(U, X)$ and from Lemma 3.2 $\tau(X) = C(\xi, X) + K(\xi, X)$. Using both relations we obtain

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, X)X, N) \quad \forall\, X \in \mathscr{S}(\zeta). \qquad (3.17)$$

Remark 3.1. *As we can see in the above proof, the compactness of the null hypersurface is only used to get $\tau(\xi) = 0$. So Proposition 3.1 remains true if the compactness assumption is dropped and $\tau(\xi) = 0$. Recall also that without compactness hypothesis if ζ is conformal then $\tau(\xi) = 0$ (see [12]). As a consequence, we get the following.*

Proposition 3.2. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a conformal vector field satisfying $\overline{\nabla}_X(d\alpha) = 0, \quad \forall\, X \in \zeta^\perp$ where $\alpha = g(\zeta, .)$. If M is totally geodesic then it holds:*

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, Y)PZ, N) \quad \forall\, X \in \mathscr{S}(\zeta). \qquad (3.18)$$

3.2. **Spatially conformally stationary symmetries and riggings.** In this section, we prove some results very usefuls for the next section. First we recall the following lemmas.

Lemma 3.3. *([1]) Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface M such that ζ is a geodesic spatially conformal stationary reference frame. Then $\tau(\xi) = \frac{\rho}{2}$.*

Lemma 3.4. *([1]) Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a closed, spatially conformal stationary reference frame.Then $\tau(X) = 0 \quad \forall X \in \mathscr{S}(\zeta)$.*

A normalized null hypersurface (M, ζ) is screen umbilic if the tensor C satisfies for all $u, v \in T_pM$, $C(p)(u, v) = \phi(p)\overline{g}(u, v)$ for some smooth function ϕ on M. We prove the following.

Proposition 3.3. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a closed and spatially conformal stationary reference frame. If M is totally umbilic with umbilic factor ϕ then (M, ζ) is screen umbilic. Moreover it holds $C = (-\frac{\rho}{2} + \phi)\overline{g}$. In particular, if M is totally geodesic then $C = -\frac{\rho}{2}\overline{g}$.*

Proof. From Lemma 3.4, we have $\tau(X) = 0 \quad \forall X \in \mathscr{S}(\zeta)$. Recall that for a closed rigging we have $\tau(X) = -C(\xi, X)$ ([12]), so that

$$C(\xi, X) = 0, \quad \forall X \in \mathscr{S}(\zeta). \tag{3.19}$$

From [12, Corollary 3.6 (4)] it holds

$$-2C(U, X) = d\alpha(U, X) + (L_\zeta\overline{g})(U, X) + \overline{g}(\zeta, \zeta)B(U, X)$$

$\forall U \in TM$ and $\forall X \in \mathscr{S}(\zeta)$. Since M is totally totally umbilic with umbilic factor ϕ and ζ is closed spatially conformally stationary, we get

$$C(X, Y) = (-\frac{\rho}{2} + \phi)\overline{g} \quad \forall X, Y \in \mathscr{S}(\zeta). \tag{3.20}$$

Since $\overline{g}(\xi, X) = 0$, from equation 3.19 and equation 3.20 it follows that $C = (-\frac{\rho}{2} + \phi)\overline{g}$.

In case M is totally geodesic, we prove the following.

Proposition 3.4. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a closed and spatially conformal stationary reference frame. If M is*

totally geodesic then it holds:

$$(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})\overline{g}(Y,PZ)=\overline{g}(\overline{R}(\xi,Y)PZ,N)\quad \forall Y,Z\in TM. \tag{3.21}$$

Proof. From Proposition 3.3, $C=-\frac{\rho}{2}\overline{g}=\lambda\overline{g}$ where we have set $\lambda=-\frac{\rho}{2}$ Now recall the following equation:

$$\overline{g}(\overline{R}(X,Y)PZ,N)=(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)+C(X,PZ)\tau(Y)-C(Y,PZ)\tau(X) \tag{3.22}$$

$\forall X,Y,Z\in TM$. Taking $X=\xi$ and using $C(\xi,PZ)=0$ (equation 3.19) and $C=\lambda\overline{g}$ we get

$$\overline{g}(\overline{R}(\xi,Y)PZ,N)=(\nabla_\xi\lambda\overline{g})(Y,PZ)-(\nabla_Y\lambda\overline{g})(\xi,PZ)-(Y,PZ)\lambda\overline{g}(Y,PZ)\tau(\xi). \tag{3.23}$$

The connection ∇ is not a metric connection but satisfies.

$$(\nabla_X g)(Y,Z)=B(X,Y)\eta(Z)+B(X,Z)\eta(Y). \tag{3.24}$$

As M is totally geodesic, we get $\nabla_X g=0,\forall\, X\in TM$. Hence (3.23) becomes

$$\overline{g}(\overline{R}(\xi,Y)PZ,N)=(\xi(\lambda)-\lambda\tau(\xi))\overline{g}(Y,PZ). \tag{3.25}$$

Since $\tau(\xi)=\frac{\rho}{2}$ (Lemma 3.3) and $\lambda=-\frac{\rho}{2}$ it follows that:

$$(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})\overline{g}(Y,PZ)=\overline{g}(\overline{R}(\xi,Y)PZ,N)\quad \forall Y,Z\in TM.$$

Proposition 3.5. *Let $(\overline{M}^{n+2},\overline{g})$ be a conformally flat Lorentzian manifold and (M,ζ) a normalized null hypersurface such that ζ is a closed and spatially conformal stationary reference frame. If M is totally geodesic then it holds:*

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=(\frac{1}{n}-\frac{1}{2})\sum_{i=1}^{n}\overline{Ric}(e_i,e_i)+(\frac{1}{2}-\frac{1}{n+1})S \tag{3.26}$$

being $(e_1,\dots,e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ and S the scalar curvature of $(\overline{M},\overline{g})$.

Proof. Since $(\overline{M},\overline{g})$ is conformally flat, the Weyl tensor vanishes. So we have:

$$\begin{aligned}\overline{R}(X,Y)Z &= -\frac{1}{n}(\overline{Ric}(X,Z)Y-\overline{Ric}(Y,Z)X+\overline{g}(X,Z)QY-\overline{g}(Y,Z)QX)\\ &\quad +\frac{S}{n(n+1)}(\overline{g}(X,Z)Y-\overline{g}(Y,Z)X).\end{aligned}$$

Take $p\in M$ and $(e_1,\dots,e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p then we get

$$\overline{R}(\xi,e_i)e_i=-\frac{1}{n}(\overline{Ric}(\xi,e_i)e_i-\overline{Ric}(e_i,e_i)\xi-Q\xi)-\frac{S}{n(n+1)}\xi$$

and

$$\overline{g}(\overline{R}(\xi, e_i)e_i, N) = -\frac{1}{n}(-\overline{Ric}(e_i, e_i) - \overline{Ric}(\xi, N)) - \frac{S}{n(n+1)}.$$

From (3.14) it follows that:

$$\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4} = \frac{1}{n}(\overline{Ric}(e_i, e_i) + \overline{Ric}(\xi, N)) - \frac{S}{n(n+1)}$$

and consequentely

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \frac{1}{n}\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + \overline{Ric}(\xi, N) - \frac{S}{n+1} \tag{3.27}$$

Finally, note that $S = \sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + 2\overline{Ric}(\xi, N)$ that is

$$\overline{Ric}(\xi, N) = -\frac{1}{2}\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + \frac{S}{2}. \tag{3.28}$$

Replacing (3.28) in (3.27) gives

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

4. Causality in conformally flat spacetimes

A pseudo-Riemannian manifold $(\overline{M}^{n+2}, \overline{g})$ is said to be (locally) conformally flat if for each point $p \in \overline{M}$, there exists an open neighborhood $\mathcal{U}$ and a positive function $e^f : \mathcal{U} \longrightarrow \mathbb{R}$ such that $\overline{g} = e^f g_0$, where $(\mathbb{E}^{n+2}, g_0)$ is the pseudo-Euclidean space. In all the paper, by conformally flat, we will always mean locally conformally flat. A necessary condition for $(\overline{M}^{n+2}, \overline{g})$ to be conformally flat is that the Weyl tensor vanish. In dimension greater or equal to 4 , this condition is sufficient as well. Many authors have investigated about conformally flat pseudo-Riemannian manifolds. In the following, we will restrict ourself on conformally flat Lorentzian manifolds which include Robertson-Walker spacetime. We put a particular attention to the causal structure of such spacetimes. We start with the following.

Theorem 4.1. *Let $(\overline{M}^{n+2}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0 \quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$is non total imprisoning.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has negative scalar curvature then $(\overline{M}, \overline{g})$ is non total imprisoning.*
3. *if $n \geq 3$ and $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is non total imprisoning.*

Proof. Suppose $(\overline{M}, \overline{g})$ is totally imprisoning. Since $(\overline{M}, \overline{g})$ is chronological, from [19, Theorem 3.9.], it contains a null line η contained in a compact minimal invariant set Ω (in the sense of [19, Definition 3.6.]) such that $\bar{\gamma} = \Omega$. Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed (embedded) achronal totally geodesic null hypersurface M [10, Theorem IV.1.]. Consider the normalized null hypersurface (M, ζ). From Proposition 3.5, it holds:

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = (\frac{1}{n} - \frac{1}{2}) \sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

Now we discuss the different cases.

1. if $n + 2 = 3$ then we have

$$\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4} = \frac{1}{2}\overline{Ric}(e_1, e_1)$$

   and by hypothesis $\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4} < 0$. Note that γ can be considered as an integral curve of ξ and since it is imprisonned in a compact set, it is defined on $\mathbb{R}$. So along γ we have

$$\frac{(\rho \circ \gamma)'}{2} - \frac{(\rho \circ \gamma)^2}{4} > 0.$$

   This yields a contradiction from the fact that γ is complete but the last differential inequality can not hold for all time (see also [12, Proof of Proposition 3.11]). Hence we conclude that $(\overline{M}, \overline{g})$ is non total imprisoning.

2. If $n + 2 = 4$ then we have

$$2(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \frac{1}{6}S$$

   and since the scalar curvature S is negative, we get the contradiction follows as in the previous case.

3. If $n \geq 3$, the hypothesis $Ric(X, X) > \frac{n-1}{(n+1)(n-2)}S$ lead to

$$(\frac{1}{n} - \frac{1}{2}) \sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S < 0$$

   and the contradiction follows as above.

The following lemma is needed in the proof of the next theorem.

Lemma 4.1. *Consider the differential equation*

$$y'(t) - y^2(t) = h(t) \tag{4.29}$$

where h is a function which is bounded below by a positive constante k. Let $]a, b[$ be the maximal interval of any solution of (4.29), then a and b are finite.

Proof. Let y be a solution of (4.29) defined on a maximal interval $]a, b[$. Suppose $b = \infty$. From (4.29), we have $\frac{y'}{y^2+k} \geq 1$. Take $t_0 \in]a, b[$, then by integrating between t_0 and t, it follows that

$$\frac{1}{\sqrt{k}} Arctan(\frac{y(t)}{\sqrt{k}}) - \frac{1}{\sqrt{k}} Arctan(\frac{y(t_0)}{\sqrt{k}}) \geq t - t_0, \forall\, t \geq t_0.$$

This means that $Arctan(\frac{y(t)}{\sqrt{k}})$ goes to ∞ as t goes to ∞, which is a contradiction. Now, suppose $a = -\infty$. Then we obtain

$$\frac{1}{\sqrt{k}} Arctan(\frac{y(t_0)}{\sqrt{k}}) - \frac{1}{\sqrt{k}} Arctan(\frac{y(t)}{\sqrt{k}}) \geq t_0 - t, \forall\, t \leq t_0.$$

This means that $Arctan(\frac{y(t)}{\sqrt{k}})$ goes to $-\infty$ as t goes to $-\infty$, which is a contradiction.

Theorem 4.2. *Let $(\overline{M}^{n+2}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ such that $div(\zeta)$ is bounded above or below. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) \leq -k \quad \forall\, X \in \zeta^{\perp}$, with k a positive constante then $(\overline{M}, \overline{g})$ is stably causal..*
2. *if $n = 2$ and the scalar curvature satisfies $S \leq -k$ with k a positive constante then $(\overline{M}, \overline{g})$ is stably causal.*
3. *if $n \geq 3$, suppose $\overline{Ric}(X, X) \geq \frac{n-1}{(n+1)(n-2)} S + k \quad \forall\, X \in \zeta^{\perp}$ with k a positive constante then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. Suppose $(\overline{M}, \overline{g})$ is not stably causal. Since it is chronological then it contains a null line ([22]). As above this null line is contained in a totally geodesic null hypersurface M and considering the normalized null hypersurface (M, ζ), (3.26) holds. Let γ be an integral curve of ξ. Then $\rho \circ \gamma$ satisfies the differential equation

$$n(\frac{y'}{2}) - n(\frac{y^2}{4}) = h(t) \tag{4.30}$$

where

$$h(t) = (\frac{1}{2} - \frac{1}{n}) \sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{n+1} - \frac{1}{2}) S.$$

By hypothesis, there exists a positive constante k such that $h \geq k$. Let $I =]a, b[$ be the maximal interval of the solution $\rho \circ \gamma$. From Lemma 4.1, a and b are finite. From (4.30) and

$h \geq k$ with $k > 0$, $\rho \circ \gamma$ is increasing. First, suppose $div(\zeta)$ is bounded above. As $\rho \circ \gamma$ is increasing, its limit at b is either infinity or some real c. But only the latter can occurs as ρ is bounded above. However, in this case the solution $\rho \circ \gamma$ is bounded near b contradicting the "theorem des bouts". We conclude that $(\overline{M}, \overline{g})$ is stably causal.
Now suppose suppose $div(\zeta)$ is bounded below. Then the limit at a of $\rho \circ \gamma$ must be finite and the contradiction follows as above.

From Theorem 4.1 and Theorem 4.2, the following holds.

Corollary 4.1. *Let $(\overline{M}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension 4 satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. If $(\overline{M}, \overline{g})$ has negative constante scalar curvature then it is non total imprisoning. Moreover if $div(\zeta)$ is bounded above or below then it is stably causal.*

For conformally stationary spacetime, we prove the following.

Theorem 4.3. *Let (M^{n+2}, g) be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a timelike conformal vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. Then the following holds:*

1. *if $n = 1$ and there exists a non negative constante k such that $\overline{Ric}(X, X) < -k$ (resp. $\overline{Ric}(X, X) > k$) $\forall X \in \zeta^{\perp}$, then $(\overline{M}, \overline{g})$ is non total imprisoning.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ and there exists a non negative constante k such that the scalar curvature satisfies $S < -k$ (resp. $S > k$) then $(\overline{M}, \overline{g})$ is non total imprisoning.*
3. *if $n \geq 3$, suppose there exists a non negative constante k such that $Ric(X, X) > \frac{n-1}{(n+1)(n-2)} S + k \quad \forall X \in \zeta$ (resp. $Ric(X, X) < \frac{n-1}{(n+1)(n-2)} S - k \quad \forall X \in \zeta^{\perp}$) then $(\overline{M}, \overline{g})$ is non total imprisoning.*

Moreover if $div(\zeta)$ is bounded above or below and k is positive then $(\overline{M}, \overline{g})$ is stably causal.

Proof. Suppose $(\overline{M}, \overline{g})$ is totally imprisoning. Then there exists a null line η contained in a smooth (topologically) closed embedded achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). From Proposition 3.2, it holds

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, Y)PZ, N), \quad \forall X \in \mathscr{S}(\zeta).$$

Following the proof of Proposition 3.5, we get

$$n(\xi(-\frac{\rho}{2})) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n}\overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

The hypothesis in each of the three case lead to $\xi(-\frac{\rho}{2}) < 0$ or $\xi(-\frac{\rho}{2}) > 0$; that is ρ is a Lyaponov function of the flow of ξ. Now consider the flow of ξ on M. Take a point $p \in \eta$ and let γ_p be the integral curve of ξ such that $\gamma_p(0) = p$. Then from [19, Theorem 3.9.] $\bar{\gamma_p} = \omega(\gamma_p)$ and it follows that p is a positively recurrent point, that is, there exists $t_n \to \infty$ such that $\gamma_p(t_n) \to p$. The contradiction follows from the fact that $\rho\circ\gamma_p$ is strictly increasing. So $(\overline{M}, \overline{g})$ is non total imprisoning. For the last part, suppose again by contradiction that $(\overline{M}, \overline{g})$ is not stably causal. Then there exists a null line η contained in a smooth (topologically) closed embedded achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). As above we either $\xi(-\frac{\rho}{2}) < -k$ or $\xi(-\frac{\rho}{2}) > k$ with k a positive constante. Recall that ξ is a $\overline{g}$-geodesic vector field and since M is (topologically) closed embedded and $\overline{M}$ null complete, then ξ is complete. Take any integral curve γ of ξ, then there exist a positive constant k such that $\rho\circ\gamma < -k$ (resp. $\rho\circ\gamma > -k$). It follows that $\rho\circ\gamma$ is onto since $\rho\circ\gamma$ is defined on whole $\mathbb{R}$, which gives the contradiction as ρ is bounded above or below.

For spacetime admitting timelike homothetic (eventually Killing) vector field, we get.

Theorem 4.4. *Let $(\overline{M}^{n+2}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a timelike homothetic (eventually Killing) vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X,X) < 0$ (resp. $\overline{Ric}(X,X) > 0$) $\forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is stably causal.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has nowhere vanishing scalar curvature then $(\overline{M}, \overline{g})$ is stably causal.*
3. *if $n \geq 3$ and $\overline{Ric}(X,X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$*
   (resp. $\overline{Ric}(X,X) < \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$) then $(\overline{M}, \overline{g})$ is stably causal.

Moreover, in each case if additionally, ζ is complete then $(\overline{M}, \overline{g})$ is causally continuous.

Proof. Suppose by contradiction that $(\overline{M}, \overline{g})$ is not stably causal. Then there exists a null line η contained in a smooth (topologically) closed embedded achronal totally geodesic

null hypersurface M. Consider the normalized null hypersurface (M,ζ) then

$$n(\xi(-\frac{\rho}{2}))=(\frac{1}{n}-\frac{1}{2})\sum_{i=1}^{n}\overline{Ric}(e_i,e_i)+(\frac{1}{2}-\frac{1}{n+1})S.$$

Since ζ is homothetic, the left hand side is zero whereas the right hand side is either positive or negative; which gives the contradiction. Hence $(\overline{M},\overline{g})$ is stably causal. Moreover is ζ is complete then $(\overline{M},\overline{g})$ is reflecting (see [14]) and then causally continuous.

Now, we consider the case when the spacetime is non chronological. In this case, the chronology violating set is $\mathcal{C}=\{x : x\ll x\}$, and is made by all the events through which there passes a closed timelike curve. The spacetime violates chronology if $\mathcal{C}\neq\emptyset$ that is if there is a closed timelike curve and it is totally vicious if $\mathcal{C}=\overline{M}$. Suppose $\mathcal{C}\neq\emptyset$, then $\mathcal{C}$ can split into equivalence classes according to Carter's equivalence relation $x\sim y\Leftrightarrow x\ll y$ and $y\ll x$. Two points belong to the same class if there is a closed timelike curve passing through them. The class of $x\in\mathcal{C}$ is denoted $[x]$. Note that $[x]=I^+(x)\cap I^-(x)$, thus $[x]$ is open. So the chronological violating set can be written $\mathcal{C}=\bigcup_\alpha\mathcal{C}_\alpha$, with $\mathcal{C}_\alpha$ its (open) connected components. The boundary of the component $\mathcal{C}_\alpha$ can be written $\partial\mathcal{C}_\alpha=\bigcup_k B_{\alpha k}$, with $B_{\alpha k}$ its (closed) connected components. Some authors has studied the compactness of the components of the chronological violating set's boundary in link with some energy condition ([15]) or absence of null line ([22]). More precisely, we have the following Kriele's theorem.

Theorem 4.5. *Suppose that $(\overline{M},\overline{g})$ satisfies the null energy condition and the null genericity condition. If a connected component of the boundary of the chronology violating set $\mathcal{C}$ is compact, then $(\overline{M},\overline{g})$ is null geodesically incomplete.*

We prove the following.

Theorem 4.6. *Let $(\overline{M}^{n+2},\overline{g})$ be a non chronological non totally vicious conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M},\overline{g})$ is null complete and admits a closed spatially conformally stationary reference frame ζ. Then the connected components of the boundary of the chronological violating set are all non compact in each of the following case:*

1. *if $n=1$ and $\overline{Ric}(X,X)<0 \quad \forall X\in\zeta^\perp$.*
2. *if $n=2$ and $(\overline{M},\overline{g})$ has negative scalar curvature.*
3. *if $n\geq 3$ and $\overline{Ric}(X,X)>\frac{n-1}{(n+1)(n-2)}S \quad \forall X\in\zeta^\perp$.*

Proof. Suppose the boundary of the chronological violating set has a compact connected component (say B) then there exists a null line η contained in B. Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M [10, Theorem IV.1.]. Consider the normalized null hypersurface (M, ζ). From Proposition 3.5, it holds:

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

Then, we discuss the different cases as in the proof of Theorem 4.1 and get the contradiction.

Totally vicious spacetimes has been of interest of research. They include Godel spacetime which is a solution of Einstein equation. It has been proved that a compact spacetime which admits a timelike conformal vector field is totally vicious. In the following, we prove that if the spacetime admits a closed spatially conformally stationary reference frame then under some curvature and completeness hypothesis it is totally vicious. More precisely, we have:

Theorem 4.7. *Let $(\overline{M}^{n+2}, \overline{g})$ be a compact conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is null complete and admits a closed spatially conformally stationary reference frame ζ. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0 \quad \forall X \in \zeta^\perp$ then $(\overline{M}, \overline{g})$ is totally vicious.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has negative scalar curvature then $(\overline{M}, \overline{g})$ is totally vicious.*
3. *if $n \geq 3$ and $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^\perp$ then $(\overline{M}, \overline{g})$ is totally vicious.*

Proof. It is well known that a compact spacetime is non chronological. Suppose $(\overline{M}, \overline{g})$ is non totally vicious, then from Theorem 4.1 the connected components of the boundary of the chronological violating set are all non compact. But, being M compact, the connected components of the boundary of the chronological violating set may be all compact, which gives the contradiction.

In case the spacetime admits an timelike affine conformal Killing vector field, we can prove the following.

Theorem 4.8. *Let $(\overline{M}^{n+2}, \overline{g})$ be a compact conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \quad \forall X \in \zeta^\perp$ where $\alpha = g(\zeta, .)$. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0$ (resp. $\overline{Ric}(X, X) > 0$) $\quad \forall X \in \zeta^\perp$ then $(\overline{M}, \overline{g})$ is totally vicious.*

2. *if* $n = 2$ *and* $(\overline{M}, \overline{g})$ *has nowhere vanishing scalar curvature then* $(\overline{M}, \overline{g})$ *is totally vicious.*
3. *if* $n \geq 3$ *and* $\overline{Ric}(X,X) > \frac{n-1}{(n+1)(n-2)}S$ *(resp.* $\overline{Ric}(X,X) < \frac{n-1}{(n+1)(n-2)}S$*)* $\forall X \in \zeta^{\perp}$ *then* $(\overline{M}, \overline{g})$ *is totally vicious.*

Proof. Note that $(\overline{M}, \overline{g})$ is complete as ζ is timelike affine conformal Killing. Suppose $(\overline{M}, \overline{g})$ is not totally vicious then as $\overline{M}$ is compact it contained a null line η (see [22], Theorem 12). Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). From Proposition 3.1 and following the proof of Proposition 3.5, we get

$$n(\xi(-\frac{\rho}{2})) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

Then, we discuss the different cases as in the proof of Theorem 4.1 and get the contradiction.

5. Quasi-Einstein spacetimes

A Lorentzian manifold is said to be quasi-Einstein ([26]) if there exists two smooth functions μ and β and a timelike unit vector field U such that its Ricci tensor satisfies:

$$\overline{Ric} = \mu\overline{g} + \beta\overline{g}(U,.)\overline{g}(U,.). \tag{5.31}$$

The notion of quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson- Walker spacetimes are quasi-Einstein manifolds. Also quasi-Einstein manifold can be taken as model of the perfect fluid spacetime in general relativity. In this section we explore causality conditions in such spacetime and finish with some applications to perfect fluide spacetime ([26]).

5.1. **Causality in quasi-Einstein spacetimes.** Let start with the following.

Lemma 5.1. *Let* $(\overline{M}, \overline{g})$ *be a Lorentzian manifold and* (M, ζ) *a normalized null hypersurface such that* ζ *is a closed and spatially conformal stationary reference frame. If* M *is totally geodesic then it holds:*

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \overline{Ric}(\xi, N) - \overline{K}(\xi, N).$$

Proof. From Proposition 3.4, it holds

$$(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})\overline{g}(Y,PZ)=\overline{g}(\overline{R}(\xi,Y)PZ,N)$$

$\forall Y,Z\in TM$. Take $p\in M$ and $(e_1,\dots,e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p then we get

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\sum_{i=1}^{n}\overline{g}(\overline{R}(\xi,e_i)e_i,N).$$

But

$$\overline{Ric}(\xi,N)=\sum_{i=1}^{n}\overline{g}(\overline{R}(\xi,e_i)e_i,N)+\overline{K}(\xi,N)$$

and hence

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\overline{Ric}(\xi,N)-\overline{K}(\xi,N).$$

Theorem 5.1. *Let $(\overline{M}^{n+2},\overline{g})$ be a quasi-Einstein Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M},\overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. If the sectional curvature satisfies $\overline{K}>\mu$ for all non degenerate plane containing ζ then $(\overline{M},\overline{g})$ is non total imprisoning.*

Proof. Suppose $(\overline{M},\overline{g})$ is totally imprisoning. Since $(\overline{M},\overline{g})$ is chronological, from [19, Theorem 3.9.], it contains a null line η contained in a compact minimal invariant set Ω (in the sense of [19, Definition 3.6.]) such that $\bar{\gamma}=\Omega$. Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M [10, Theorem IV.1.]. Consider the normalized null hypersurface (M,ζ). From Lemma 5.1, we have

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\overline{Ric}(\xi,N)-\overline{K}(\xi,N).$$

From (5.31), $\overline{Ric}(\xi,\xi)=\beta(\overline{g}(U,\xi))^2$. As M is totally geodesic, $\overline{Ric}(\xi,\xi)=\beta(\overline{g}(U,\xi))^2=0$. This follows from $\overline{Ric}(\xi)=\xi(H)+\tau(\xi)H-|\overset{*}{A}_\xi|^2$. Since $\overline{g}(U,\xi)$ never vanishes, β vanishes on M. As $\overline{g}(\xi,N)=1$, we find that $\overline{Ric}(\xi,N)=\mu$ and then

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\mu-\overline{K}(\xi,N). \tag{5.32}$$

Using the hypothesis on the sectional curvature, we get

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})<0.$$

The contradiction follows as in the proof of Theorem 4.1.

Theorem 5.2. *Let $(\overline{M}^{n+2}, \overline{g})$ be a quasi-Einstein Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ such that $div(\zeta)$ is bounded above or below. If there exists a positive constante k such that $\overline{K} \geq \mu + k$ for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. Suppose $(\overline{M}, \overline{g})$ is not stably causal. Then it contains a null line. As above this null line is contained in a totally geodesic null hypersurface M and considering the normalized null hypersurface (M, ζ), it holds

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \mu - \overline{K}(\xi, N).$$

Let γ be an integral curve of ξ. Then $\rho \circ \gamma$ satisfies the differential equation

$$n(\frac{y'}{2}) - n\frac{y^2}{4}) = h(t) \tag{5.33}$$

where

$$h(t) = \mu - \overline{K}(\xi, N).$$

By hypothesis, $h \geq k > 0$ and following the proof of Theorem 4.2 we get the contradiction.

As in the conformally flat case, we prove the following.

Theorem 5.3. *Let $(\overline{M}^{n+2}, \overline{g})$be a quasi-Einstein Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a timelike conformal vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. If $\overline{K} > \mu$ (resp. $\overline{K} < \mu$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is non total imprisoning. Moreover if $div(\zeta)$ is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \mu + k$ (resp. $\overline{K} \leq \mu - k$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal and $(\overline{M}, \overline{g})$ is causally continuous if ζ is complete.*

Proof. Suppose $(\overline{M}, \overline{g})$ is totally imprisoning. We know that there exists a null line η contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). From Proposition 3.1, it holds

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, Y)PZ, N)$$

$\forall X \in \mathscr{S}(\zeta)$. Take $p \in M$ and $(e_1, \ldots, e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p then we get

$$n(\xi(-\frac{\rho}{2})) = \sum_{i=1}^{n} \overline{g}(\overline{R}(\xi, e_i)e_i, N).$$

But

$$\overline{Ric}(\xi, N) = \sum_{i=1}^{n} \overline{g}(\overline{R}(\xi, e_i)e_i, N) + \overline{K}(\xi, N)$$

and hence

$$n(\xi(-\frac{\rho}{2})) = \overline{Ric}(\xi, N) - \overline{K}(\xi, N).$$

As $\overline{g}(\xi, N) = 1$, we find that $\overline{Ric}(\xi, N) = \mu$ and then

$$n(\xi(-\frac{\rho}{2})) = \mu - \overline{K}(\xi, N).$$

The hypothesis on the sectional curvature lead to $\xi(-\frac{\rho}{2}) < 0$ or $\xi(-\frac{\rho}{2}) > 0$ that is ρ is a Lyaponov function of the flow of ξ. The contradiction follows from the existence of a recurrent point (see proof of Theorem 4.3). So $(\overline{M}, \overline{g})$ is non total imprisoning. If $div(\zeta)$ is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \mu + k$ (resp. $\overline{K} \leq \mu - k$) for all non degenerate plane containing ζ, then $\xi(-\frac{\rho}{2}) < -k$ (resp. $\xi(-\frac{\rho}{2}) > k$) and the contradiction follows as in the proof of Theorem 4.3. Hence $(\overline{M}, \overline{g})$ is stably causal. If additionally ζ is complete then $(\overline{M}, \overline{g})$ is reflecting ([14]) and then causally continuous.

5.2. **Physical model: perfect fluid spacetimes.**

Definition 5.1. ([24]) *A perfect fluid on a spacetime $(\overline{M}^4, \overline{g})$ is a triple (U, ρ, p) where :*

1. *U is a timelike future-pointing unit vector field on $\overline{M}$ called the flow vector field.*
2. *ρ is the energy density function; p is the pressure function.*
3. *The stress-energy tensor is $T = (\rho + p)U^* \otimes U^* + p\overline{g}$, where U^* is the one-form metrically equivalent to U.*

Let $(\overline{M}, \overline{g})$ be a perfect fluid spacetime satisfying the Einstein equation (with cosmological constant Λ). Then it holds:

$$\overline{Ric} + (\Lambda - \frac{1}{2}S)\overline{g} = (\rho + p)U^* \otimes U^* + p\overline{g},$$

where S is the scalar curvature. It follows that

$$\overline{Ric} = (\frac{1}{2}S - \Lambda + p)\overline{g} + (\rho + p)U^* \otimes U^*.$$

Hence $(\overline{M}, \overline{g})$ is quasi-Einstein. Note that $(\overline{M}, \overline{g})$ satisfies the null energy condition if and only if $\rho + p \geq 0$. From Theorem 5.1 and Theorem 5.2, we can state.

Theorem 5.4. *Let $(\overline{M}^4, \overline{g})$ be a perfect fluid spacetime satisfying the Einstein equation (with cosmological constant Λ). Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. If $\rho + p \geq 0$ and the sectional curvature satisfies $\overline{K} > \frac{1}{2}S - \Lambda + p$ for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is non total imprisoning. Moreover if div(ζ) is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \frac{1}{2}S - \Lambda + p + k$ for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal.*

From Theorem 5.3, we have also the following.

Theorem 5.5. *Let $(\overline{M}^4, \overline{g})$ be a perfect fluid spacetime satisfying the Einstein equation (with cosmological constant Λ). Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a time-like conformal vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. If $\rho + p \geq 0$ and the sectional curvature satisfies $\overline{K} > \frac{1}{2}S - \Lambda + p$ (resp. $\overline{K} < \frac{1}{2}S - \Lambda + p$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is non total imprisoning. Moreover if div(ζ) is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \frac{1}{2}S - \Lambda + p + k$ (resp.$\overline{K} \leq \frac{1}{2}S - \Lambda + p - k$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal and $(\overline{M}, \overline{g})$ is causally continuous if ζ is complete.*

6. Causality in Hubble isotropic spacetimes

Following ([16]) we state:

Definition 6.1. *An ordered triple $(\overline{M}, \overline{g}, \zeta)$ is called Hubble-isotropic spacetime if $(\overline{M}, \overline{g})$ is a spacetime together with a future-directed reference frame ζ, and the shear and the acceleration of ζ vanish, i.e ζ is a geodesic spatially conformal stationary reference frame.*

Obviously, the notion of Hubble-isotropic spacetimes do naturally include conformally stationary and stationary ones with vanishing acceleration. The following theorem due to A. Dirmeier [7] gives the form of the Lorentzian metric in Hubble-isotropic spacetime of splitting type.

Theorem 6.1. *Let $(\overline{M} = \mathbb{R} \times F^{n+1}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime of splitting type. Then there are two positive functions A, s on $\overline{M}$ and a Riemannian metric h on F, such that $\zeta = \frac{1}{A}\partial_t$ and the metric is given by*

$$\begin{aligned}\overline{g}(t,x) &= -A^2(t,x)dt^2 + 2pr_2^*(b(t,x)) \vee dt + s^2(t,x)pr_2^*(h_x) \\ &\quad - \frac{pr_2^*(b(t,x)) \otimes pr_2^*(b(t,x))}{A^2(t,x)}\end{aligned}$$

with $x \in F, t = pr_1 : \mathbb{R} \times F \to \mathbb{R}, pr_2 : \mathbb{R} \times F \to F$ and $(b_t)_t \in \mathbb{R}$ a family of one-forms on F obeying

$$b_{(t,x)} = A(t,x)(\beta_x + \int_{t_0}^{t} \mathcal{H}(dA)_{(t',x)} dt')$$

for some $t_0 \in \mathbb{R}$ and a one-form β on F and $\mathcal{H}(dA)$ satisfies

$$dA = (\partial_t A)dt + \mathcal{H}(dA).$$

The expansion Θ of ζ is given by $\Theta = \overline{div}(\zeta) = \frac{(n+1)(\partial_t s)(t,x)}{A(t,x)s(t,x)}$.

These spacetimes are of particular interest in physics, especially in cosmology and are special cases of shear-free cosmological models ([13]). Nevertheless, their global properties have scarcely been analyzed up to now. The standard references for Hubble-isotropic spacetimes are ([16]) and ([17]). We explore some causality aspects of such spacetimes and prove the following.

Lemma 6.1. *Let $(\overline{M}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime (with ζ complete) and (ϕ_t) be the flow of ζ.*

1. *If the expansion Θ is non negative and γ is a causal curve (resp. a timelike curve) then $\forall\, s \leq 0, \phi_s \circ \gamma$ is also a causal curve (resp. a timelike curve).*
2. *If Θ is non positive and γ is a causal curve (resp. a timelike curve) then $\forall\, s \geq 0, \phi_s \circ \gamma$ is also a causal curve (resp. a timelike curve).*

Proof. Recall that for a timelike reference ζ we have

$$(L_\zeta \overline{g}) = 2\sigma - 2u \vee \dot{u} + \frac{2}{n}\Theta h,$$

where $u = \overline{g}(\zeta, .), \dot{u} = \overline{g}(\overline{\nabla}_\zeta \zeta, .), \sigma$ is the shear tensor and $h = \overline{g} + u \otimes u$. As the shear and the acceleration vanish, we get $L_\zeta \overline{g} = \frac{2}{n}\Theta h$. We know also that $L_\zeta \overline{g} = \lim_{t \to 0}(\frac{1}{t}[\phi_t^* \overline{g} - \overline{g}])$. Let (ϕ_t) be a flow of ζ. Let v be a tangent vector at a point p, and set $w = d\phi_s(v)$ for all s. Hence

$$\lim_{t \to 0} \frac{1}{t}[\overline{g}(d\phi_t(w), d\phi_t(w)) - \overline{g}(w,w)] = \frac{2}{n}\Theta(\overline{g}(w,w) + \overline{g}^2(\zeta, w)).$$

Since $\phi_s \circ \phi_t = \phi_{s+t}$, it holds

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t}[\overline{g}(d\phi_{s+t}(v), d\phi_{s+t}(v)) - \overline{g}(d\phi_s(v), d\phi_s(v))] \quad &= \quad \frac{2}{n}\Theta(\overline{g}(d\phi_s(v), d\phi_s(v)) \\ &\quad + \overline{g}^2(\zeta, d\phi_s(v))). \end{aligned} \tag{6.34}$$

Note that $\overline{g}(d\phi_s(v), d\phi_s(v)) + \overline{g}^2(\zeta, d\phi_s(v)) \geq 0$. In fact this holds trivially if $d\phi_s(v)$ is spacelike or null and by the reverse Cauchy-Schwartz inequality, it holds also if $d\phi_s(v)$ is

timelike. Hence if ϕ is non negative (resp. non positive) then (6.34) means that the real-valued function $s \longmapsto \overline{g}(d\phi_s(v), d\phi_s(v))$ has non negative (resp. non positive) derivative. So if Θ is non negative (resp. non positive) then $\forall\, s \leq 0, \overline{g}(d\phi_s(v), d\phi_s(v)) \leq \overline{g}(v, v)$ (resp. $\forall\, s \geq 0, \overline{g}(d\phi_s(v), d\phi_s(v)) \leq \overline{g}(v, v)$). In particular if Θ is non negative and γ is a causal curve (resp. a timelike curve) then $\forall\, s \leq 0, \phi_s \circ \gamma$ is also a causal curve (resp. a timelike curve). In the same way, if Θ is non positive and γ is a causal curve (resp. a timelike curve) then $\forall\, s \geq 0, \phi_s \circ \gamma$ is also a causal curve (resp. a timelike curve).

In ([14]) it is proved that a conformally stationary spacetime with a complete stationary vector field is reflecting. We prove similar result for Hubble-isotropic spacetime with non positive (resp. non negative) expansion.

Theorem 6.2. *Let $(\overline{M}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime with non negative (resp. non positive) expansion. If ζ is complete then $(\overline{M}, \overline{g})$ is past reflecting (resp. future reflecting).*

Proof. We suppose that the expansion is non negative and show past reflectivity that is $I^+(p) \supseteq I^+(q) \Longrightarrow I^-(p) \subseteq I^-(q)$ (the non positive case is similar). Take any $p \neq q$ in $\overline{M}$ and let $\phi_t : \overline{M} \longrightarrow \overline{M}$ be the flow of ζ at the stage $t \in \mathbb{R}$. Assuming the first inclusion, it is enough to prove $p_{-\epsilon} := \phi_{-\epsilon}(p) \in I^-(q)$, for all $\epsilon > 0$ (notice that the relation $\ll$ is open and then $r \ll p$ will lie also in $I^-(p_{-\epsilon})$ for small ϵ). As $q_\epsilon := \phi_\epsilon(q) \in I^+(p)$, there exists a future directed timelike curve γ joining p and q_ϵ. From Lemma 6.1, $\phi_{-\epsilon} \circ \gamma$ is also a timelike curve and this curve connects $p_{-\epsilon}$ and q as required.

Definition 6.2. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold.*

1. *A function $f : \overline{M} \longrightarrow \mathbb{R}$ is a generalized time function if $\forall\, p, q \in \overline{M}, p < q \Rightarrow f(p) < f(q)$.*
2. *A function f is a semi-time function if f is continuous and strictly increasing on future directed timelike curve.*

Remark 6.1. *It is known that past reflectivity (resp. future reflectivity) is equivalent to the continuity of the volume function t^- (resp. t^+) (* [4, Proposition 3.21]*). Moreover t^- (resp. t^+) is strictly increasing on any future-directed timelike curve if and only if $(\overline{M}, \overline{g})$ is chronological (*[20]*).*

As a consequence we have.

Corollary 6.1. *Let $(\overline{M}, \overline{g}, \zeta)$ be a chronological Hubble-isotropic spacetime with non negative (resp. non positive) expansion. If ζ is complete then the volume functions t^- and t^+ of $(\overline{M}, \overline{g})$ are semi-time functions.*

In ([23]) the author gave the following characterization of distinguishing and strongly causal spacetimes.

Theorem 6.3.

1. *The spacetime $(\overline{M}, \overline{g})$ is future (resp. past) distinguishing if and only if for every $x, z \in \overline{M}, (x, z) \in J^+$ and $x \in \overline{J^+(z)}$ imply $x = z$ (resp. $(x, z) \in J^+$ and $z \in \overline{J^-(x)}$ imply $x = z$).*
2. *The spacetime $(\overline{M}, \overline{g})$ is strongly causal if and only if for every $x, z \in \overline{M}, (x, z) \in J^+$ and $(z, x) \in \overline{J^+}$ imply $x = z$*

We prove the following.

Theorem 6.4. *Let $(\overline{M}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime with non positive (resp. non negative) expansion. If $(\overline{M}, \overline{g})$ admits a generalized time function and ζ is complete then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. We consider the case the expansion is non negative (the non positive case is analogous). Suppose $(\overline{M}, \overline{g})$ is not distinguishing. Then from Theorem 6.3, there exists two distinct points $x, z \in \overline{M}$ such that $(x, z) \in J^+$ and $x \in \overline{J^+(z)}$. Since x and z are distinct and $(x, z) \in J^+$ we have $f(x) < f(z)$. Also, since $x \in \overline{J^+(z)}$, there exists a sequence $(x_n)_n$ converging to x such that $\forall\, n, x_n \in J^+(z)$. Let ϕ_t denote the flow of ζ at the stage t and γ_x the integral curve of ζ such that $\gamma_x(0) = x$. As f is a generalized time function,

$$f \circ \gamma_x : \mathbb{R} \longrightarrow \mathbb{R}$$

is strictly increasing and so continuous outside a countable set. Let $t_0 \in \mathbb{R}, t_0 < 0$ such that $f \circ \gamma_x$ is continuous at t_0. From ([25], Proposition A.1) f is continuous at $\gamma_x(t_0) = \phi_{t_0}(x)$. As ϕ_{t_0} maps a causal curve to a causal curve (Lemma 6.1) and $(x, z) \in J^+$, we have $(\phi_{t_0}(x), \phi_{t_0}(z)) \in J^+$ so that $f(\phi_{t_0}(x)) < f(\phi_{t_0}(z))$. Moreover as $\forall\, n, x_n \in J^+(z)$, it holds $\forall\, n, \phi_{t_0}(z) \in J^+(\phi_{t_0}(x_n))$ and then $f(\phi_{t_0}(z)) < f(\phi_{t_0}(x_n))$. Since f is continuous at $\phi_{t_0}(x)$ this lead to $f(\phi_{t_0}(z)) \leq f(\phi_{t_0}(x))$; which gives the contradiction. We conclude that $(\overline{M}, \overline{g})$ is future distinguishing. We show similarly that $(\overline{M}, \overline{g})$ is past distinguishing. Hence $(\overline{M}, \overline{g})$ is distinguishing. So the volume time function t^+ and t^- are generalized time function (see [20]). Since ζ is complete then past reflectivity or future reflectivity hold on $(\overline{M}, \overline{g})$ (Theorem 6.2).

From Remark 6.1, t^- or t^+ is continuous and then t^+ or t^- is a time function, that is $(\overline{M}, \overline{g})$ is stably causal.

Now we put attention to conformally stationnary spacetime. Locally, such a spacetime with a timelike conformally Killing vector field K can be written as a standard conformally stationary spacetime with respect to K, i.e., a product manifold $\overline{M} = \mathbb{R} \times S$ and the metric can be written as

$$\overline{g}(t, x) = \Omega(t, x)(-\beta(x)dt^2 + 2\omega_x dt + h_x), \tag{6.35}$$

being Ω a positive function on $\overline{M}$, and h, β, ω, respectively a Riemannian metric, a positive function and a 1-form, all on S. The case $\Omega 1$, or independent of t, corresponds to a standard stationary spacetime. Then, a natural question is to wonder when a spacetime admitting a (necessarily complete) conformally stationary timelike vector field K can be written globally as above. A positive answer is given in [14]. Precisely the authors prove the following.

Theorem 6.5. *Let $(\overline{M}, \overline{g})$ be a spacetime which admits a complete conformally stationary vector field K. Then, it admits a standard splitting (6.35) if and only if $(\overline{M}, \overline{g})$ is distinguishing. Moreover, in this case, $(\overline{M}, \overline{g})$ is causally continuous.*

In the following, we prove that the standard splitting holds if the distinction property is replaced by the existence of a generalized time function. Note that this is a weak condition than being distinguishing since any distinguishing spacetime admits a generalized time function. More precisely we prove:

Theorem 6.6. *Let $(\overline{M}, \overline{g})$ be a spacetime which admits a complete conformastationary vector field K. Then, it admits a standard splitting (6.35) if and only if $(\overline{M}, \overline{g})$ admits a generalized time function. Moreover, in this case, $(\overline{M}, \overline{g})$ is causally continuous.*

Proof. Suppose $(\overline{M}, \overline{g})$ admits a standard splitting. From [14, Theorem 3.2], it is known that $(\overline{M}, \overline{g})$ is causally continuous. Hence it admits a time function. Conversely, suppose $(\overline{M}, \overline{g})$ admits a generalized time function. As K is timelike conformal, there exists a conformal metric g^* to $\overline{g}$ such that ζ is Killing for $(\overline{M}, g^*)$ and $g^*(\zeta, \zeta) = -1$. Hence K is geodesic and $(\overline{M}, g^*, \zeta)$ is a Hubble-isotropic spacetime (with vanishing expansion). From Theorem 6.3, $(\overline{M}, g^*)$ is distinguishing and so is $(\overline{M}, \overline{g})$. From Theorem 6.5, $(\overline{M}, \overline{g})$ admit a standard splitting and is causally continuous.

Theorem 6.7. *Let $(\overline{M}, \overline{g}, \zeta)$ be a chronological Hubble-isotropic spacetime with positive (resp. negative) expansion. If ζ is complete then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. Suppose $(\overline{M},\overline{g})$ is not strongly causal. Then from Theorem 6.3, there exists two distinct points $x,z\in\overline{M}$ such that $(x,z)\in J^+$ and $(z,x)\in\overline{J^+}$. Since $(z,x)\in\overline{J^+}$, there exists two sequences $(x_n)_n$ and $(z_n)_n$ converging respectively to x and z such that $\forall\, n, x_n\in J^+(z_n)$. We consider first the case the expansion is non negative (the non positive case is analogous). As $(x,z)\in J^+$, there exists a future directed causal curve γ joining x and z. The curve γ is a null curve otherwise z will be contained in $I^+(x)$ and since $(z,x)\in\overline{J^+}$, $(\overline{M},\overline{g})$ would contained a closed timelike curve in contradiction with the chronological assumption. $\forall s\leq 0, \phi_s\circ\gamma$ is a causal curve. Suppose that $\forall\, s\leq 0, \phi_s\circ\gamma$ is a null curve then the real-valued function $s\longmapsto \overline{g}(d\phi_s(\gamma'),d\phi_s(\gamma'))$ vanishes identically on $(-\infty,0)$. The contradiction follows from the fact that its derivative is $\frac{2}{n}\Theta[\overline{g}(d\phi_s(\gamma'),d\phi_s(\gamma'))+\overline{g}^2(\zeta,d\phi_s(\gamma')]$ (6.34), which is nowhere zero as Θ never vanishes and $d\phi_s(\gamma')$ is lightlike. So there exists $s_0<0$ such that $\phi_{s_0}\circ\gamma$ is a causal curve with timelike part which means that $\phi_{s_0}(z)\in I^+(\phi_{s_0}(x))$. Using $(z,x)\in\overline{J^+}$ and Lemma 6.1, we get also $(\phi_{s_0}(z),\phi_{s_0}(x))\in\overline{J^+}$. This contradicts again the chronological assumption. Hence $(\overline{M},\overline{g})$ is strongly causal and in particular distinguishing. Since ζ is complete then past reflectivity or future reflectivity hold on $(\overline{M},\overline{g})$ (Theorem 6.2). So $(\overline{M},\overline{g})$ is stably causal.

References

[1] C. Atindogbe, M. Gutiérrez and R. Hounnonkpe, *topologically closed null hypersurfaces in Lorentzian manifold*, preprint.

[2] C. Atindogbe, Blaschke type normalization on lightLike Hypersurfaces, *Journal of Mathematical Physics, Analysis, Geometry.* **6** No. 4 (2010) 362-382.

[3] C. Atindogbe and H. T. Fosting, Newton transformations on null hypersurfaces, *Communications in Mathematics.* **23** (2015) 57-83.

[4] J. K. Beem, and P. E. Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker Inc., New York. 1981.

[5] J. K. Beem, Conformal changes and geodesic completeness, *Commun. Math. Phys.* **49** (1976) 179-186.

[6] M. C. Chaki and R. K. Maity, *On quasi-Einstein manifolds*, Publ. Math. Debrecen 57 (2000), no. 3-4, 297–306.

[7] A. Dirmeier, *Particular timelike flows in global Lorentzian geometry,* Technische Universität Berlin, Fakultät II - Mathematik und Naturwissenschaften. Ph. D thesis (2012).

[8] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic, 364, Dordrecht, 1996.

[9] K. L. Duggal, *A Review on Metric Symmetries Used in Geometry and Physics* International Scholarly Research Network, ISRN Geometry, Volume 2012, Article ID 983403, 26 pages doi:10.5402/2012/983403.

[10] G. J. Galloway, Maximum principles for null hypersurfaces and null splitting theorems, *Ann. Henri Poincaré.* **1** (2000) 543-567

[11] E. García-Río and D.N. Kupeli, Singularity versus splitting theorems for stably causal spacetimes, *Annals Glob. Anal. Geom.* **14** (1996) 301-312.

[12] M. Gutiérrez and B. Olea, Induced Riemannian structures on null hypersurfaces, *Math. Nachr.* **289** (2015) 1219-1236.

[13] M. Gürses, M. Plaue and M. Scherfner *On a particular type of product manifolds and shear-free cosmological models*, Class. Quantum Grav. **28** (2011), 175009.

[14] M. A. Javaloyes and M. Sánchez, *A note on the existence of standard splittings for conformally stationary spacetimes*, Classical Quantum Gravity, 25 (2008) no.16, 168001,7.

[15] Kriele, M. *The structure of chronology violating sets with compact closure*, Class. Quantum Grav. **6**, 1607–1611 (1989)

[16] W. Hasse and V. Perlick,*On spacetimes with an isotropic Hubble law*, Class. Quant. Grav. **16** (1999), 2559-2576.

[17] W. Hasse, Geometrische Auswertungsmethoden für kosmologische Beobachtungen, Ph.D. thesis, TU Berlin (in German), 1991.

[18] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.

[19] E. Minguzzi, Non-imprisonment conditions on spacetime, *J. Math. Phys.* **49** (2008) 062503-1-062503-9.

[20] E. Minguzzi and M. Sánchez, The causal hierarchy of spacetimes. Recent developments in pseudo-Riemannian geometry, 299–358, ESI Lect. Math. Phys., *Eur. Math. Soc.*, Zürich, 2008.

[21] E. Minguzzi, *Weak distinction and the optimal definition of causal continuity*, Class. Quant. Grav. **25** (2008), 075015.

[22] E. Minguzzi, *Chronological spacetimes without lightlike lines are stably causal*, Commun. Math. Phys. **288** (2009), 801–819.

[23] E. Minguzzi, *The causal ladder and the strength of K-causality. I*, Class. Quant. Grav. **25** (2008), 015009.

[24] B. O'Neill, *Semi-Riemannian geometry with Application to Relativity*, Academic Press, 1983.

[25] A. Reinnie, Ben E. Whale, *Generalised time functions and finiteness of the Lorentzian distance*, Journal of Geometry and Physics **106** (2016), 108-121.

[26] A. A. Shaikh, Y. H. Kim and S. K. Hui, *On Lorentzian quasi-Einstein manifolds*, J. Korean Math. Soc. 48 (2011), No. 4, pp. 669-689

Department of Mathematics, University of Abomey-Calavi (Benin)

Email address: atincyr@gmail.com

Institut de Mathématiques et de Sciences Physiques, University of Abomey-Calavi (Benin)

Email address: rhounnonkpe@ymail.com

International Journal of Maps in Mathematics

Volume:1, Issue:2, (2018), Pages:219-236

ISSN: 2636-7467 (Online)

www.journalmim.com

EXISTENCE OF GLOBAL ATTRACTOR FOR A MODEL OF SUSPENSION BRIDGE

FATIMA ZOHRA MAHDI, ALI HAKEM*, AND MOHAMED FERHAT

Abstract. The goal of this paper is to establish a well-posedness result and the existence of a finite-dimensional global attractor for the following model of a suspension bridge equations:

$$u_{tt} + \Delta^2 u - \Delta u + u_t - \int_0^\infty \mu(s)\Delta^2 u(x,y,t-s)\ ds + h(u) = f, \quad \text{in} \quad \Omega \times \mathbb{R}^+.$$

Furthermore, the regularity of global attractor is achieved. This results extend previous works

1. Introduction

From the physics point of view, the suspension bridge equation describes the transverse deflection of the roadbed in the vertical plane. The suspension bridge equations were presented by A.C. Lazer and P.J. McKenna[1] as new problems in the field of nonlinear analysis. Lately, similar models have been studied by many authors, but most of them have only concentrated on the existence of solutions, (see for instance [7,8] and the references therein), while the existence of the global attractors for the suspension bridge equations are most of our concern.

Received:2018-06-01 *Revised:2018-08-31* *Accepted:2018-09-15*

*2010 Mathematics Subject Classification:*34B15, 74B20,35C10, 35G31.

Key words: Suspension bridge, global attractors

* Corresponding author

Recently, S. Liu and Q. Ma [4] studied the long time dynamical behavior for the following extensible suspension bridge equations with past history

$$u_{tt} + u_t + \Delta^2 u + (\alpha - \beta\|\nabla u\|^2_{L^2(\Omega)})\Delta u - \int_\Omega \mu(s)\Delta^2 u(t-s)ds + ku^+ = g(x) \quad in \quad \Omega \times \mathbb{R}^+.$$

They proved the existence of the global attractors by using the contraction function method and the regularity. We point out here that C.K. Zhong and Q. Ma [9] proved the existence of strong solutions and global attractors for the suspension bridge equations. Related to this subject, we can mention the work of J.Y. Park and J.R. Kang [10,11]. In that papers, they obtained the existence of pullback attractor for the non autonomous suspension bridge equations and the existence of global attractors for the suspension bridge equations with nonlinear damping.

The recent work of A. Ferrero and F. Gazzola [2], suggested a rectangular plate model describing the displacement of a suspension bridge in the downward direction. The plate $\Omega = (0, \pi) \times (-l, l)$ is assumed to be partially hinged on the vertical edges

$$u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \quad \forall y \in (-l, l),$$

and free on the horizontal edges

$$u_{yy}(x, \pm l) + \sigma u_{xx}(x, \pm l) = u_{yyy}(x, \pm l) + (2 - \sigma)u_{xxy}(x, \pm l) - u_y(x, \pm l) = 0, \quad \forall x \in (0, \pi).$$

They established the well-posedness and discussed several other stationary problems. We also recall the results by S.A. Messaoudi et al ([5,6]), where the authors investigated the following problem

$$u_{tt} + \Delta^2 u + h(u(x, y, t)) + \delta_1 u_t(x, y, t) + \delta_2 u_t(x, y, t - \tau) = f, \; in \quad \Omega \times (0, \infty),$$

which describes the downward displacement of a suspension bridge in the presence of a hanger restoring force $h(u)$ and external force f which includes gravity and a delay term which accounts for its history. They proved the existence of a finite-dimensional global attractor. For more details on suspension bridge models, we refer the reader to the new Book on mathematical models for suspension bridges by F. Gazzola [3]. Motivated by the previous works, in the present paper we investigate the problem (1.1) in which we contribute

to the results obtained in the cited references.

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u + u_t - \int_0^\infty \mu(s)\Delta^2 u(x,y,t-s)ds \\ +h(u) = f, \quad in\ \Omega \times \mathbb{R}^+, \\ u(0,y,t) = u_{xx}(0,y,t) = 0, \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, \quad for\ (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) - u_y(x,\pm l,t) = 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u(x,y,t) = u_0(x,y), u_t(x,y,0) = u_1(x,y), \quad in\ \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,\pi) \times (-l,l) \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$. The memory kernel $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is an absolutely continuous function which may possibly blows up at 0.

2. Preliminaries

We present the following conditions about memory kernel

$$(H_1) : \mu \in \mathcal{C}^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu'(s) \le 0 \le \mu(s), \quad \forall s \in \mathbb{R}^+,$$

$$(H_2) : l = 1 - \int_0^\infty \mu(s)ds = 1 - \mu_0 > 0, \quad \forall s \in \mathbb{R}^+,$$

$$(H_3) : \mu'(s) + \delta\mu(s) \le 0, \quad \forall s \in \mathbb{R}^+, \quad \delta > 0.$$

Concerning the forcing term $h : \mathbb{R} \to \mathbb{R}$, we assume that

$$(G_1) : h(0) = 0, \quad \text{and} \quad |h(u) - h(v)| \le K_0 \left(1 + |u|^p + |v|^p\right) |u - v|, \quad \forall u, v \in \mathbb{R}, \tag{2.2}$$

where $K_0 > 0$ and $p > 0$. Condition $p > 0$ implies that $H_*^2(\Omega) \hookrightarrow L^{2(p+1)}(\Omega)$. In addition, we assume that

$$-K_1 \le H(u) \le h(u)u, \quad \forall u \in \mathbb{R}. \tag{2.3}$$

As in C.M. Dafermos [12], we introduce the relative displacement past history function as

$$\phi^t(x,y,s) = u(x,y,t) - u(x,y,t-s), \quad (x,y,s) \in \Omega \times \mathbb{R}^+, \quad t \ge 0. \tag{2.4}$$

Notice that ϕ satisfies the equation

$$\phi_t^t + \phi_s^t - u_t = 0, \tag{2.5}$$

with the boundary condition

$$\phi^t(x,y,0) = 0,$$

and the initial condition

$$\phi^0(x,y,s) = u_0(x,y) - u(x,y,-s) := w(s),$$

where w represents the history of u. Consequently, problem (1.1) becomes

$$\begin{cases} u_{tt} + \left(1 - \int_0^\infty \mu(s)ds\right)\Delta^2 u - \Delta u + u_t \\ + \int_0^\infty \mu(s)\Delta^2\phi^t(s)ds + h(u) = f \quad in\ \Omega\times(0,\infty), \\ \phi_t^t + \phi_s^t - u_t = 0, \quad in\ \Omega\times(0,\infty), \end{cases} \tag{2.6}$$

with the boundary conditions

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = 0 \quad for\ (y,t)\in(-l,l)\times\mathbb{R}^+, \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 \quad for\ (y,t)\in(-l,l)\times\mathbb{R}^+, \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, \quad (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) - u_y(x,\pm l,t) = 0, \quad (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ \phi^t(0,y,s) = \phi^t_{xx}(0,y,s) = 0, \quad for\ (y,s)\in(-l,l)\times\mathbb{R}^+, \\ \phi^t(\pi,y,s) = \phi^t_{xx}(\pi,y,s) = 0, \quad for\ (y,s)\in(-l,l)\times\mathbb{R}^+, \\ \phi^t_{yy}(x,\pm l,s) + \sigma\phi^t_{xx}(x,\pm l,s) = 0, \quad (x,s)\in(0,\pi)\times\mathbb{R}^+, \\ \phi^t_{yyy}(x,\pm l,s) + (2-\sigma)\phi^t_{xxy}(x,\pm l,s) - \phi^t_y(x,\pm l,t) = 0, \quad (x,s)\in(0,\pi)\times\mathbb{R}^+, \end{cases} \tag{2.7}$$

and the initial conditions

$$\begin{cases} u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \qquad in \quad \Omega \\ \\ \phi^0(x,y,s) = \phi_0(x,y,s) = u_0(x,y) - u(x,y,-s), \qquad in \quad \Omega\times[0,+\infty). \end{cases} \tag{2.8}$$

We will use the standard functional space and denote $(.,.)$ be a $L^2(\Omega)$- inner product and $\|.\|_p$ be $L^p(\Omega)$ norm. Especially, we take

$$H = V_0 = L^2(\Omega), \quad V = V_1 = H^2_*(\Omega),$$

with

$$H^2_*(\Omega) = \Big\{\xi\in H^2(\Omega), \xi = 0\ on\ \{0,\pi\}\times\{-l,l\}\Big\},$$

equipped with respective inner product and norm,

$$(u,v) = (\Delta u, \Delta v), \quad \|u\|_V = \|\Delta u\|_2.$$

Define

$$D(A) = \Big\{u\in H^4(\Omega) \quad \text{such that} \quad (2.7) \quad \text{holds}\Big\},$$

where $Au = \Delta^2 u$, and equip this space with the inner product (Au, Av), and the norm $\|Au\|_2^2 = (Au, Au)$. We have the following continuous dense injections

$$D(A) \subset V \subset H = H^* \subset V^*,$$

where H^*, V^* are the dual spaces of H, V respectively.

We consider the relative displacement ϕ as a new variable, we introduce the weighted L^2-space

$$L^2_\mu(\mathbb{R}^+, V_i) = \Big\{\phi : \mathbb{R}^+ \longrightarrow V_i \quad \text{such that} \quad \int_0^\infty \mu(s)\|\phi(s)\|^2_{V_i} ds < \infty\Big\},$$

which is a Hilbert space endowed with inner product and norm

$$(u, v)_{\mu, V_i} = \int_0^\infty \mu(r)(u(r), v(r))_{V_i} dr,$$

$$\|u\|^2_{\mu, V_i} = (u, u)_{\mu, V_i} = \int_0^\infty \mu(r)\|u(r)\|^2_{V_i} dr, \quad i = 0, 1, 2,$$

respectively, where $V_2 = D(A^{\frac{3}{4}})$ and $V_3 = D(A)$. Finally, we introduce the following Hilbert spaces

$$\mathcal{H}_0 = V \times H \times L^2_\mu(\mathbb{R}^+; V), \quad \mathcal{H}_1 = D(A) \times V \times L^2_\mu(\mathbb{R}^+; D(A)),$$

equipped with the norms

$$\|u, u_t, \phi\|_{\mathcal{H}_0} = \|\Delta u\|_2^2 + \|u_t\|_2^2 + \|\phi^t\|^2_{\mu, V},$$

and

$$\|u, u_t, \phi\|_{\mathcal{H}_1} = \|\nabla \Delta u\|_2^2 + \|\nabla u_t\| + \|\phi^t\|^2_{\mu, D(A^{\frac{3}{4}})}.$$

Using the Poincaré inequality we obtain

$$\lambda_1 \|v\|_2^2 \leq \|\Delta v\|_2^2, \quad \forall v \in V,$$

where λ_1 denotes the first eigenvalue of $\Delta^2 v = \lambda v$ in Ω.

In order to obtain the global attractors of the problem (2.6)-(2.8), we need the following theorem. The well-posedness of problem (2.6)-(2.8) can be obtained by Faedo-Galerkin method (see[13]) and combining with a prior estimate of 3.1, we omit and only give the following theorem.

Theorem 2.1. *Assume that assumptions $(H_1) - (H_3)$, (G_1) hold and $f \in L^2(\Omega)$. Problem (2.6)-(2.8) has a weak solution $(u, u_t, \phi) \in C([0, T], \mathcal{H}_0)$ with initial data $(u_0, u_1, \phi^0) \in \mathcal{H}_0$, satisfying*

$$u \in L^\infty(0, T; V), \quad u_t \in L^\infty(0, T; H), \quad \phi \in L^\infty(0, T, L^2_\mu(\mathbb{R}^+, V)),$$

and the mapping $\{u_0, u_1, \phi^0\} \to \{u(t), u_t(t), \phi^t\}$ *is continuous in* $\mathcal{H}_0$*. In addition, if* $z^i(t) = (u^i(t), u^i_t(t), \phi^i)$ *is a weak solution of problem (2.6)-(2.8) corresponding to initial data* $z^i(0) = (u^i_0, u^i_1, \phi^i_0), \quad i = 1, 2$*, then one has*

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_0} \le e^{ct}\|z_1(0) - z_2(0)\|_{\mathcal{H}_0}, \quad t \in [0, T],$$

for some constant $c \ge 0$*.*

The well-posedness of problem (2.6)-(2.8) implies that the family of operator $S(t) : \mathcal{H}_0 \to \mathcal{H}_0$ defined by

$$S(t)(u_0, u_1, \phi^0) = (u(t), u_t(t), \phi^t), \quad t \ge 0,$$

where $(u(t), u_t(t), \phi^t)$ is the unique weak solution of the problem (2.6)-(2.8), satisfies the semigroup properties and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on $\mathcal{H}_0$. Now, we recall some fundamentals of theory of infinite dimensional systems in mathematical physics. These abstract results will be used in our consideration.

Definition 2.1. *A dynamical system* $(\mathcal{H}, S(t))$ *is dissipative if it possesses a bounded absorbing set, that is, a bounded set* $\mathfrak{B} \subset \mathcal{H}$ *such that for any bounded set* $B \subset \mathcal{H}$ *there exists* $t_B \ge 0$ *satisfying*

$$S(t)B \subset \mathfrak{B}, \quad \forall t \ge t_B.$$

Definition 2.2. *Let* X *be Banach space and* B *a bounded subset of* X*. We call a function* $\Phi(.,.)$ *which is defined on* $X \times X$ *a contractive function on* $B \times B$ *if for any sequence* $\{x_n\}_{n=1}^{\infty} \subset B$*, there is a subsequence* $\{x_{nk}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$*, such that*

$$\lim_{k\to\infty} \lim_{l\to\infty} \Phi(x_{nk}, x_{nl}) = 0. \tag{2.9}$$

Denote all such contractive functions on $B \times B$ *by* $\mathfrak{C}$*.*

Definition 2.3. *Let* $\{S(t)\}_{t\ge0}$ *be a semi-group on a Banach space* $(X, \|.\|)$ *that has a bounded absorbing set* B_0*. Moreover, assume that for* $\epsilon > 0$ *there exist* $T = T(B_0, \epsilon)$ *and* $\Phi_T(.,.) \in \mathfrak{C}(B_0)$ *such that*

$$\|S(T)x - S(T)y\| \le \epsilon + \Phi_T(x, y), \forall (x, y) \in B_0,$$

where Φ_T *depends on* T*. Then* $\{S(t)\}_{t\ge0}$ *is asymptotically compact in* X*, i.e , for any bounded sequence* $\{y_n\}_{n=1}^{\infty} \subset X$ *and* $\{t_n\}$ *with* $t_n \to \infty$*,* $\{S(t_n)y_n\}_{n=1}^{\infty}$ *is precompact in* X*.*

Theorem 2.2. *[14] A dissipative dynamical system* $(\mathcal{H}, S(t))$ *has a compact global attractor if and only if it is asymptotically smooth.*

Our main result in the following

Theorem 2.3. *Assume that assumptions $(H_1)-(H_3)$ and (G_1) are fulfilled. Let $h \in C^1(\mathbb{R}, \mathbb{R})$ and $f \in L^2(\Omega)$ be given. Then the dynamical system $(\mathcal{H}_0, S(t))$ corresponding to the system (2.6) − (2.8) has a compact global attractor $\mathcal{A} \subset \mathcal{H}_0$, which attracts any bounded set in $\mathcal{H}_0$ with $\|.\|_{\mathcal{H}_0}$.*

3. Global attractor in $\mathcal{H}_0$

In order to prove Theorem 2.3, we will apply the abstract results presented in Section 2. The first step is to show that the dynamical system $(\mathcal{H}_0, S(t))$ is dissipative. The second step is to verify the asymptotic compactness. Then the existence of compact global attractor is guaranteed by Theorem 2.2.

3.1. **A Priori estimates in $\mathcal{H}_0$.** First, taking the scalar product in H of the first equation of (2.6) with $v = u_t + \theta u$, after a computation, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + 2\int_\Omega H(u)dx - 2\int_\Omega fudx\right) + \theta l\|\Delta u\|_2^2 + \theta\|\nabla u\|_2^2 \\ &+(1-\theta)(u_t, v) + (\phi^t, u_t)_{\mu,v} + \theta(\phi^t, u)_{\mu,V} + \theta\int_\Omega h(u)udx - \theta\int_\Omega fudx = 0.\end{aligned} \tag{3.10}$$

Exploiting $(H_1) - (H_3)$ and Hölder inequality, we have

$$(1-\theta)(u_t, v) = (1-\theta)\|v\|_2^2 - \theta(1-\theta)(u, v),$$

$$\begin{aligned}(\phi^t, u_t)_{\mu,V} &= (\phi^t, \phi^t_t + \phi^t_s)_{\mu,V} = \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \int_0^\infty \mu(s)(\phi^t, \phi^t_s(s))_V ds \\ &= \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\|\phi^t\|^2_V ds \\ &= \frac{1}{2}\frac{d}{dt}\|\phi^t\|_{\mu,V} - \frac{1}{2}\int_0^\infty \mu'(s)\|\phi^t(s)\|^2_V ds \\ &\geq \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \frac{\delta}{2}\int_0^\infty \mu(s)\|\phi^t\|^2_V ds = \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,V} + \frac{\delta}{2}\|\phi^t\|^2_{\mu,V},\end{aligned} \tag{3.11}$$

$$\theta(\phi^t, u)_{\mu,V} \geq -\frac{\delta}{4}\|\phi^t\|^2_{\mu,V} - \frac{(1-l)\theta^2}{\delta}\|\Delta u\|_2^2.$$

We choose θ small enough, such that

$$1 - \frac{(1-l)\theta}{\delta} - \frac{\theta}{2\lambda_1} \geq 1-\theta, \quad \frac{1}{2} - \theta \geq \frac{1}{4},$$

then combining with Hölder, Young and Poincaré inequalities, we obtain

$$
\begin{aligned}
&\theta l\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\Delta u\|_2^2+(1-\theta)\|v\|_2^2-\theta(1-\theta)(u,v)\\
&\geq \theta\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\Delta u\|_2^2+(1-\theta)\|v\|_2^2-\frac{\theta}{\sqrt{\lambda_1}}\|\Delta u\|_2\|v\|_2\\
&\geq \theta l\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\Delta u\|_2^2+(1-\theta)\|v\|_2^2-\left(\frac{\theta^2}{2\lambda_1}\|\Delta u\|_2^2+\frac{1}{2}\|v\|_2^2\right)\\
&= \theta l\left(1-\frac{(1-l)\theta}{\delta l}-\frac{\theta}{2\lambda_1 l}\right)\|\Delta u\|_2^2+\left(\frac{1}{2}-\theta\right)\|v\|_2^2\\
&\geq \theta l(1-\theta)\|\Delta u\|_2^2+\frac{1}{4}\|v\|_2^2.
\end{aligned}
\tag{3.12}
$$

Collecting with (3.11) and (3.12), there holds

$$
\begin{aligned}
&\frac{d}{dt}\left(\|v\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}^2+2\int_\Omega H(u)dx-2\int_\Omega fudx\right)\\
&+\frac{1}{2}\|v\|_2^2+2\theta l(1-\theta)\|\Delta u\|_2^2+2\theta\|\nabla u\|_2^2+\frac{\delta}{2}\|\phi^t\|_{\mu,V}^2\\
&+2\theta\int_\Omega h(u)udx-2\theta\int_\Omega fudx\leq 0.
\end{aligned}
\tag{3.13}
$$

Provided that $\theta_0=\min\left\{2\theta l\left((1-\theta)-\frac{1}{4}\right),2\theta,\frac{1}{4},\frac{\delta}{2}\right\}$, let

$$
E(t)=\|v\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}^2+2\int_\Omega H(u)dx-2\int_\Omega fudx,
\tag{3.14}
$$

and

$$
I(t)=\|v\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}^2+2\int_\Omega h(u)udx-2\int_\Omega fudx.
\tag{3.15}
$$

We have

$$
\frac{d}{dt}E(t)+\theta_0 I(t)\leq 0,
\tag{3.16}
$$

which implies

$$
E(t)\leq -\theta_0\int_0^t I(\tau)d\tau+E(0),
\tag{3.17}
$$

where

$$
E(0)=\|u_1+\theta u_0\|_2^2+l\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+\|\phi^0\|_{\mu,V}^2+2\int_\Omega H(u_0)dx-2\int_\Omega fu_0dx.
\tag{3.18}
$$

Noticing that (2.3)and (3.14)-(3.15), and using the compact Sobolev embedding theorem we get

$$
E(t)\geq \|v\|_2^2+\left(l-\frac{\lambda_1+2\theta_0}{2\lambda_1}\right)\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}-M_1.
\tag{3.19}
$$

Similarly

$$
I(t)\geq \|v\|_2^2+\left(l-\frac{\lambda_1+2\theta_0}{2\lambda_1}\right)\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\phi^t\|_{\mu,V}-M_1,
\tag{3.20}
$$

where $M_1 = \dfrac{2}{\lambda_1}\|f\|_2^2 + K_1|\Omega|$. Therefore, let $\dfrac{\lambda_1 + 2\theta_0}{2\lambda_1} < l$, and $0 < \theta_0 < \lambda_1(l - \frac{1}{2})$, we have

$$l - \frac{\lambda_1 + 2\theta_0}{2\lambda_1} > 0. \tag{3.21}$$

Associated with (3.19)-(3.20), there exists a positive constant C_1 such that

$$E(t) \geq C_1 \left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1, \tag{3.22}$$

$$I(t) \geq C_1 \left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1. \tag{3.23}$$

So we deduce from (3.22)-(3.23) and (3.17) that

$$\begin{aligned} &C_1 \left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1 \leq \\ &-\theta_0 \int_0^t [C_1 \left(\|v\|_2^2 + l\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V}^2\right) - M_1]dt + E(0). \end{aligned} \tag{3.24}$$

Thus, for any $\rho_1^2 > \frac{M_1}{C_1}$, there exists $t_0 = t_0(B)$ such that

$$\|v(t_0)\|_2^2 + l\|\Delta u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2 + \|\phi^t(t_0)\|_{\mu,V} \leq \rho_1^2, \tag{3.25}$$

and we end up to.

Lemma 3.1. *Assume that assumptions $(H_1) - (H_3)$ and (G_1) hold and $h \in C(\mathbb{R},\mathbb{R}), f \in L^2(\Omega)$, then the ball of $\mathcal{H}_0$, $B_0 = B_{\mathcal{H}_0}(0,\rho_1)$, centered at 0 of radius ρ_1, is an absorbing set in $\mathcal{H}_0$ for the group $S(t)$. For any bounded subset B in $\mathcal{H}_0$, $S(t)B \subset B_0$ for $t \geq t_0$. There exists a positive constant $\mu_1 > \rho_1$ such that*

$$\|\Delta u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\phi^t\|_{\mu,V} \leq \mu_1^2, \quad \forall t \geq t_0. \tag{3.26}$$

3.2. **Existence of global attractor.** First we prove an important Lemma.

Lemma 3.2. *Under the hypotheses of Theorem 2.3, there exists a constant $\mu_2 > \rho_1$, such that*

$$\|\nabla\Delta u\|_2^2 + \|\nabla u_t\|_2^2 + \|\phi^t\|_{\mu,D(A^{\frac{3}{4}})} \leq \mu_2^2, \quad \forall t \geq t_0. \tag{3.27}$$

Proof. Multiplying $(2.6)_1$ by $-\Delta\varsigma = -\Delta u_t - \theta\Delta u$ and integrating over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2\right) + \theta l\|\nabla\Delta u\|_2^2 + \theta\|\Delta u\|_2^2 \\ &+ (1-\theta)(u_t, -\Delta\varsigma) + (\phi^t, u_t)_{\mu,D(A^{\frac{3}{4}})} + \theta(\phi^t, u)_{\mu,D(A^{\frac{3}{4}})} \\ &+ (h(u), -\Delta\varsigma) + (f, \Delta\varsigma) = 0. \end{aligned} \tag{3.28}$$

Similar to previous estimates, we see that

$$(1-\theta)(u_t, -\Delta\varsigma) = (1-\theta)\|\nabla\varsigma\|_2^2 - \theta(1-\theta)(\nabla u, \nabla\varsigma),$$

$$(\phi^t, u_t)_{\mu,D(A^{\frac{3}{4}})} \geq \frac{1}{2}\frac{d}{dt}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} + \frac{\delta}{2}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})},$$

and

$$\theta(\phi^t, u)_{\mu,D(A^{\frac{3}{4}})} \geq -\frac{\delta}{4}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} - \frac{(1-l)\theta^2}{\delta}\|\nabla\Delta u\|_2^2.$$

Whereupon

$$\begin{aligned}&\theta\left(1-\frac{(1-l)\theta}{\delta l}\right)\|\nabla\Delta u\|_2^2 + (1-\theta)\|\nabla\varsigma\|_2^2 - \theta(1-\theta)(\nabla u, \nabla\varsigma)\\ &\geq \theta(1-\theta)\|\nabla\Delta u\|_2^2 + \frac{1}{4}\|\nabla\varsigma\|_2^2.\end{aligned}$$

Then we get from (3.28)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2 + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}\right) + \theta l(1-\theta)\|\nabla\Delta u\|_2^2\\ &+\frac{1}{4}\|\nabla\varsigma\|_2^2 + \theta\|\Delta u\|_2^2 + \frac{\delta}{4}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} \leq (h(u)-f, \Delta\varsigma).\end{aligned} \tag{3.29}$$

Similarly, exploiting the bound $\|u\|_2^2 \leq c$, which implies that $\|h(u)\|^2_{L^\infty} \leq c$ and

$$(h(u)-f, \Delta u_t + \theta\Delta u) \leq (\|h(u)\|^2_{L^\infty} + \|f\|_2^2)(\|\Delta u_t\|_2^2 + \|\Delta u\|_2^2) \leq c. \tag{3.30}$$

So, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2 + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}\right) + 2\theta(1-\theta)\|\nabla\Delta u\|_2^2\\ &+2\theta l\|\nabla\Delta u\|_2^2 + \frac{1}{2}\|\nabla\varsigma\|_2^2 + 2\theta\|\Delta u\|_2^2 + \frac{\delta}{2}\|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})} \leq 2c.\end{aligned} \tag{3.31}$$

Thus, denote

$$F(t) = l\|\nabla\Delta u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\Delta u\|_2^2 + \|\phi^t\|^2_{\mu,D(\frac{3}{4})}.$$

We easily get

$$\frac{d}{dt}F(t) + \theta_0 F(t) \leq \tilde{C},$$

where $\theta_0 = \min\{2\theta(1-\theta), 2\theta, \frac{1}{2}, \frac{\delta}{2}\}$, $\quad \tilde{C} = 2c$. By the Gronwall Lemma, we get

$$F(t) \leq e^{-\theta_0 t}F(0) + \frac{\tilde{C}}{\theta_0}.$$

Using the fact that $F(t) \geq \|\nabla\Delta u\|_2^2 + \|\nabla u_t\|_2^2 + \|\phi^t\|^2_{\mu,D(A^{\frac{3}{4}})}$, then (3.27) holds. Next we show an essential inequality to prove Theorem 2.3 .

Lemma 3.3. *Under the hypotheses of Theorem 2.3, given a bounded set $B \subset \mathcal{H}_0$, let $z_1 = (u, u_t, \phi)$ and $z_2(t) = (v, v_t, \xi)$ be two weak solutions of problem (2.6)-(2.8) such that $z_1(0) = (u_0, u_1, \phi^0)$ and $z_2(0) = (v_0, v_1, \xi^0)$ are in B. Then, we have $\forall t \geq 0$*

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{H}_0} \leq e^{-\nu_1 t}\|z_1(0) - z_2(0)\|^2_{\mathcal{H}_0} + C_3\int_0^t e^{-\nu_1(t-s)}\|u(s) - v(s)\|^2_{2(p+1)}ds, \tag{3.32}$$

where $\nu_1 > 0$ is a small constant and p, C_3 are positive constants.

Proof. Let us fix a bounded set $B \subset \mathcal{H}_0$. We set $w = u - v$ and $\zeta = \phi - \xi$. Then (w, ζ) satisfy

$$\begin{cases} w_{tt} + l\Delta^2 w - \Delta w + w_t + \displaystyle\int_0^\infty \mu(s)\Delta^2\zeta(s)ds + h(u) - h(v) = 0, \\ \zeta_t = -\zeta_s + w_t, \end{cases} \tag{3.33}$$

with initial conditions

$$w(0) = u_0 - v_0, w_t(0) = u_1 - v_1, \zeta^0 = \phi_0 - \xi_0.$$

Taking the scalar product in H of $(3.33)_1$ with $\varsigma = w_t + \theta w$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2\right) + \theta l\|\Delta w\|_2^2 + \theta\|\nabla w\|_2^2 + (1-\theta)(w_t, \varsigma) \\ &+(\zeta^t, w_t)_{\mu,V} + \theta(\zeta^t, w)_{\mu,V} + (h(u) - h(v), \varsigma) = 0. \end{aligned} \tag{3.34}$$

Combining with the previous discussion, we can obtain

$$(1-\theta)(w_t, \varsigma) = (1-\theta)\|\varsigma\|_2^2 - \theta(1-\theta)(w, \varsigma),$$

$$(\zeta^t, w_t)_{\mu,V} \geq \frac{1}{2}\frac{d}{dt}\|\zeta^t\|_{\mu,V}^2 + \frac{\delta}{2}\|\zeta^t\|_{\mu,V}^2,$$

and

$$\theta(\zeta^t, w)_{\mu,V} \geq -\frac{\delta}{4}\|\zeta^t\|_{\mu,V} - \frac{(1-l)\theta^2}{\delta}\|\Delta w\|_2^2.$$

We have

$$\begin{aligned} &\theta l\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|\Delta w\|_2^2 + (1-\varsigma)\|\varsigma\|_2^2 - \theta(1-\theta)(w, \varsigma) \\ &\geq \theta l(1-\theta)\|\Delta w\|_2^2 + \frac{1}{4}\|\varsigma\|_2^2. \end{aligned}$$

Then we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2 + \|\zeta^t\|_{\mu,V}\right) + \theta l(1-\theta)\|\Delta w\|_2^2 \\ &+\frac{1}{4}\|\varsigma\|_2^2 + \theta\|\nabla w\|_2^2 + \frac{\delta}{4}\|\zeta^t\|_{\mu,V}^2 \leq -(h(u) - h(v), \varsigma). \end{aligned} \tag{3.35}$$

We use (2.2) and Young's inequality to obtain

$$\begin{aligned} &\left|-\int_\Omega h(u) - h(v)(w_t + \theta w)dx\right| \leq K_0\int_\Omega (1 + |u|^p + |v|^p)\,|w||w_t + \theta w|dx \\ &\leq K_0\int_\Omega \left(|\Omega|^{\frac{p}{2(p+2)}} + \|u\|_{2(p+1)}^p + \|v\|_{2(p+1)}^p\right)\|w\|_{2(p+1)}(\|w_t\|_2^2 + \theta\|w\|_2^2) \\ &\leq \left(\frac{K_0^2 c_B}{\theta} + \frac{2\theta K_0 c_B}{\lambda_1}\right)\|w\|_{2(p+1)}^2 + \frac{\theta}{4}\|\varsigma\|_2^2. \end{aligned} \tag{3.36}$$

In above inequality, we have used the fact that $\|w_t\|_2^2 = \|\varsigma - \theta w\|_2^2$ and $c_B > 0$ is an embedding constant for $L^{2(p+1)}(\Omega) \hookrightarrow L^2(\Omega)$. Integrating (3.33), we get from (3.35)

$$\begin{aligned}
&\frac{d}{dt}\left(l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2 + \|\zeta^t\|_{\mu,V}^2\right) + 2\theta l(1-\theta)\|\Delta w\|_2^2 \\
&+ \left(\frac{1}{2} - \frac{\theta}{2}\right)\|\varsigma\|_2^2 + 2\theta\|\nabla w\|_2^2 + \frac{\delta}{2}\|\zeta^2\|_{\mu,V}^2 \\
&\leq \left(\frac{K_0^2 c_B}{\theta} + \frac{\theta 2K_0 c_B}{\lambda_1}\right)\|w\|_{2(p+1)}^2.
\end{aligned} \tag{3.37}$$

Choosing θ small enough, such that

$$2\theta(1-\theta) > 0, \quad \frac{1}{2} - \frac{\theta}{2} > 0.$$

Thus, if we denote

$$W(t) = l\|\Delta w\|_2^2 + \|\varsigma\|_2^2 + \|\nabla w\|_2^2 + \|\zeta^t\|_{\mu,V}^2,$$

then we easily find

$$\frac{d}{dt}W(t) + \nu_1 W(t) \leq C_3\|w\|_{2(p+1)}^2,$$

where $\nu_1 = \min\left\{2\theta(1-\theta), \frac{1-\theta}{2}, \frac{\delta}{2}\right\}$, $C_3 = \dfrac{K_0^2 c_B}{\delta} + \dfrac{\theta 2K_0 c_B}{\lambda_1}$ which implies that

$$W(t) \leq e^{-\nu_1 t}W(0) + C_3\int_0^t e^{-\nu_1(t-s)}\|w\|_{2(p+1)}^2 ds.$$

Invoking $W(t) \geq \|z_1(t) - z_2(t)\|_{\mathcal{H}_0}^2$, we deduce (3.32).

Lemma 3.4. *Under assumptions of Theorem 2.3, the dynamical system $(\mathcal{H}_0, S(t))$ corresponding to the problem (2.6)-(2.8) is asymptotically smooth.*

Proof. Let B be a bounded subset of $\mathcal{H}_0$ positively invariant with respect to $S(t)$. Denote by C_B serval positive constants that are dependent on B but not on t. For $z_0^1, z_0^2 \in B$, $S(t)z_0^1 = (u(t), u_t(t), \phi^t)$ and $S(t)z_0^1 = (v(t), v_t(t), \xi^t)$ are the solutions of (2.6)-(2.8). Then given $\epsilon > 0$, from inequality (3.27), we can choose $T > 0$ such that

$$\|S(t)z_0^1 - S(t)z_0^2\|_{\mathcal{H}_0} \leq \epsilon + C_B\left(\int_0^T \|u(s) - v(s)\|_{2(p+1)}^2 ds\right)^{\frac{1}{2}}, \tag{3.38}$$

where $C_B > 0$ is a constant which depends only on the size of B. The condition $p > 0$ implies that $2 < 2(p+1) < \infty$. Taking $\varrho = \frac{1}{2}(1 - \frac{1}{p+1})$ and applying Gagliardo-Nirenberg interpolation inequality, we have

$$\|u(t) - v(t)\|_{2(p+1)} \leq C\|\Delta(u(t) - v(t))\|_2^{\varrho}\|u(t) - v(t)\|_2^{1-\varrho}.$$

Since $\|\Delta u(t)\|_2$ and $\|\Delta v(t)\|_2$ are uniformly bounded, there exists a constant $C_B > 0$ such that

$$\|u(t) - v(t)\|_{2(p+1)}^2 \leq C_B \|u(t) - v(t)\|_2^{2(1-\varrho)}. \tag{3.39}$$

Then, from (3.38) and (3.39) we obtain

$$\|S(t)z_0^1 - S(t)z_0^2\|_{\mathcal{H}_0} \leq \epsilon + \Phi_T(z_0^1, z_0^2),$$

with

$$\Phi_T(z_0^1, z_0^2) = C_B \left(\int_0^T \|u(s) - v(s)\|_2^{2(1-\varrho)} ds \right)^{\frac{1}{2}}.$$

The following proof $\Phi_T \in \mathfrak{C}$ namely Φ_T satisfies (2.9). Indeed , give a sequence $(z_0^n) = (u_0^n, u_1^n, \phi_0^n) \in B$, let us write $S(t)(z_0^n) = (u^n(t), u_t^n(t), \phi^{n,t})$ is uniformly bounded in $\mathcal{H}_0$. On the other hand,(u^n, u_t^n) is bounded in $C([0,T], V \times H), T > 0$.

By the compact embedding $V \subset H$, the Aubin lemma implies that there exists a subsequence (u^{nk}) that converges strongly in $C([0,T], H)$. Thus,

$$\lim_{k \to \infty} \lim_{l \to \infty} \int_0^T \|u^{nk}(s) - u^{nl}\|_2^{2(1-\varrho)} ds = 0.$$

Then (2.9) holds.

Proof of Theorem 2.3. Lemma 3.1 and Lemma 3.3 imply that $(\mathcal{H}_0, S(t))$ is a dissipative dynamical system which is asymptotically smooth. Then it has compact global attractor from theorem 2.2.

4. Asymptotic regular estimates

Theorem 4.1. *Under assumptions of Theorem 2.3, then the global attractor $\mathcal{A}$ is a bounded subset of $\mathcal{H}_1$.*

In order to prove Theorem 4.1, we fix a bounded set $B \subset \mathcal{H}_0$ and for $z = (u_0, u_1, \phi^0) \in B$, we split the solution $S(t)z = (u(t), u_t(t), \phi^t)$ of problem (2.6)-(2.8) into the sum

$$S(t)z = D(t)z + K(t)z,$$

where $D(t)z = z_1(t)$ and $K(t)z = z_2(t)$, namely $z = (u, u_t, \phi^t) = z_1 + z_2$. Furthermore,

$$u = v + w, \quad \phi^t = \zeta^t + \xi^t, \quad z_1 = (v, v_t, \zeta^t), \quad z_2 = (w, w_t, \xi^t),$$

where $z_1(t)$ satisfies for $(x,t) \in (0,\pi) \times \mathbb{R}^+$ and $(x,s) \in (0,\pi) \times \mathbb{R}^+$

$$\begin{cases} v_{tt} + l\Delta^2 v - \Delta v + v_t + \displaystyle\int_0^\infty \mu(s)\Delta^2 \zeta^t(s)ds = 0, \\ \zeta_t^t = -\zeta_s^t + v_t, \\ v(0,y,t) = v_{xx}(0,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ v(\pi,y,t) = v_{xx}(\pi,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ v_{yy}(x,\pm l,t) + \sigma v_{xx}(x,\pm l,t) = 0, \quad for\ (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ v_{yyy}(x,\pm l,t) + (2-\sigma) v_{xxy}(x,\pm l,t) - v_y(x,\pm l,t) = 0, \\ \zeta^t(0,y,s) = \zeta_{xx}^t(0,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \zeta^t(\pi,y,s) = \zeta_{xx}^t(\pi,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \zeta_{yy}^t(x,\pm l,s) + \sigma \zeta_{xx}^t(x,\pm l,s) = 0, \quad for\ (x,s) \in (0,\pi) \times \mathbb{R}^+, \\ \zeta_{yyy}^t(x,\pm l,s) + (2-\sigma)\zeta_{xxy}^t(x,\pm l,s) - \zeta_y^t(x,\pm l,t) = 0, \\ v(x,y,\tau) = u_\tau(x,y), \quad \zeta^\tau(x,y,s) = \phi_\tau(x,y,s). \end{cases} \tag{4.40}$$

And $z_2(t)$ satisfies for $(x,t) \in (0,\pi) \times \mathbb{R}^+$ and $(x,s) \in (0,\pi) \times \mathbb{R}^+$

$$\begin{cases} w_{tt} + l\Delta^2 w - \Delta w + w_t + \displaystyle\int_0^\infty \mu(s)\Delta^2 \xi^t(s)ds + h(u) = f, \\ \xi_t^t = -\xi_s^t + w_t, \\ w(0,y,t) = w_{xx}(0,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ w(\pi,y,t) = w_{xx}(\pi,y,t) = 0 \quad for\ (y,t) \in (-l,l) \times \mathbb{R}^+, \\ w_{yy}(x,\pm l,t) + \sigma w_{xx}(x,\pm l,t) = 0, \quad for\ (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ w_{yyy}(x,\pm l,t) + (2-\sigma) w_{xxy}(x,\pm l,t) - w_y(x,\pm l,t) = 0, \\ \xi^t(0,y,s) = \xi_{xx}^t(0,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \xi^t(\pi,y,s) = \xi_{xx}^t(\pi,y,s) = 0, \quad for\ (y,s) \in (-l,l) \times \mathbb{R}^+, \\ \xi_{yy}^t(x,\pm l,s) + \sigma \xi_{xx}^t(x,\pm l,s) = 0, \quad for\ (x,s) \in (0,\pi) \times \mathbb{R}^+, \\ \xi_{yyy}^t(x,\pm l,s) + (2-\sigma)\xi_{xxy}^t(x,\pm l,s) - \xi_y^t(x,\pm l,t) = 0, \\ w(x,y,\tau) = 0, \quad \xi^\tau(x,y,s) = \xi_\tau(x,y,s) = 0. \end{cases} \tag{4.41}$$

The well-posedness of the problem (4.40) and (4.41) can be obtained by Faedo-Galerkin method. Furthermore, combining with a priori estimate of 3.1, about the solution $z_1(t)$ of equation (4.40), we have the following result:

Lemma 4.1. *Under assumptions of Theorem 2.3, there exists a constant $k_0 > 0$, such that the solution of* (4.40) *satisfies the following inequality*

$$\|D(t)z\|_{\mathcal{H}_0}^2 \leq Ce^{-k_0 t},$$

where C is a constant.

About the solution of equation (4.41), we have the following results:

Lemma 4.2. *Under the assumptions of Theorem 2.3, there exists a constant $N > 0$, such that the solution of (4.41) satisfies the inequality bellow*

$$\|K(t)z\|_{\mathcal{H}_1}^2 \leq N.$$

Proof. Taking the scalar product in H of $(4.41)_1$ with $A\varsigma = Aw_t + \theta Aw$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2\right) + \theta l\|Aw\|_2^2 + \theta\|\nabla\Delta w\|_2^2 \\ &+(1-\theta)(w_t, Aw) + (\xi^t, w_t)_{\mu,D(A)} + \theta(\xi^t, w)_{\mu,D(A)} + (h(u), A\varsigma) = (f, A\varsigma).\end{aligned} \tag{4.42}$$

Similar to the previous discussion, there yields

$$\begin{aligned}(1-\theta)(w_t, A\varsigma) &= (1-\theta)\|\Delta\varsigma\|_2^2 - \theta(1-\theta)(Aw, \varsigma), \\ (\xi^t, w_t)_{\mu,D(A)} &\geq \frac{1}{2}\frac{d}{dt}\|\xi^t\|_{\mu,D(A)}^2 + \frac{\delta}{2}\|\xi^t\|_{\mu,D(A)}^2, \\ \theta(\xi^t, w)_{\mu,D(A)} &\geq -\frac{\delta}{4}\|\xi^t\|_{\mu,D(A)}^2 - \frac{(1-l)\theta^2}{\delta}\|Aw\|_2^2.\end{aligned}$$

Then, we get from (4.42)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|_{\mu,D(A)}^2\right) \\ &+\theta l\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|Aw\|_2^2 + (1-\theta)\|\Delta\varsigma\|_2^2 + \theta\|\nabla\Delta w\|_2^2 + \frac{\delta}{4}\|\xi^t\|_{\mu,D(A)}^2 \\ &-\theta(1-\theta)(Aw, \varsigma) + (h(u), A\varsigma) = (f, A\varsigma).\end{aligned} \tag{4.43}$$

We have

$$\begin{aligned}&\theta l\left(1 - \frac{(1-l)\theta}{\delta l}\right)\|Aw\|_2^2 + (1-\theta)\|\Delta\varsigma\|_2^2 - \theta(1-\theta)(Aw, \varsigma) \\ &\geq \theta l(1-\theta)\|Aw\|_2^2 + \frac{1}{4}\|\Delta\varsigma\|_2^2.\end{aligned} \tag{4.44}$$

By Lemma 3.1 and the Sobolev embedding theorem we know that $h(u)$, $h'(u)$ are uniformly bounded in L^∞ that there exists a constant $K_3 > 0$, such that

$$|h(u)| \leq K_3, and \quad |h'(u)| \leq K_3.$$

Combining with the Hölder, Young and Cauchy and (3.26), (3.27), it follows that

$$\begin{aligned}(h(u), A\varsigma) \quad &= \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) \geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) \\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - (h'(u)u_t, Aw) \\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - \int_\Omega h'(u)|u_t||Aw|dx \\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - K_3\mu_1\|Aw\|_2 \\ &\geq \frac{d}{dt}(h(u), Aw) + \theta(h(u), Aw) - \frac{\theta l}{4}\|Aw\|_2^2 - \frac{K_3^2\mu_1^2}{\theta l},\end{aligned} \tag{4.45}$$

and

$$(f, A\varsigma) = (f, Aw_t + \theta Aw) = \frac{d}{dt}(f, Aw) + \theta(f, Aw). \tag{4.46}$$

Thus, collecting (4.44)-(4.46) from (4.43) yields

$$\begin{aligned}
&\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} + 2(h(u), Aw) - 2(f, Aw)\right)\\
&+2l\left(\theta(1-\theta) - \frac{\theta}{4}\right)\|Aw\|_2^2 + \frac{1}{2}\|\Delta\varsigma\|_2^2 + 2\theta\|\nabla\Delta w\|_2^2 + \frac{\delta}{2}\|\xi^t\|^2_{\mu,D(A)}\\
&+2\theta(h(u), Aw) - 2\theta(f, Aw) \le \frac{K_3^2\mu_1^2}{\theta l}.
\end{aligned} \tag{4.47}$$

Taking $\theta_0 = \min\Big\{2\theta(1-\theta) - \frac{\theta}{2}, 2\theta, \frac{\delta}{2}, \frac{1}{2}\Big\}$ we can obtain from (4.47)

$$\begin{aligned}
&\frac{d}{dt}\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} + 2(h(u), Aw) - 2(f, Aw)\right)\\
&+\theta_0\left(l\|Aw\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} + 2(h(u), Aw) - 2(f, Aw)\right)\\
&\le \frac{K_3^2\mu_1^2}{\theta l}.
\end{aligned} \tag{4.48}$$

On the other hand, by the Hölder inequality, the Sobolev embedding theorem and (3.26), it follows that

$$\begin{aligned}
&\frac{d}{dt}\|\frac{l}{2}\|Aw\|_2^2 + 2(h(u), Aw)\|_2^2 \ge \frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u)\|_2^2\\
&-\frac{4}{l}\int_\Omega |h(u)|.|h'(u)||u_t|dx \ge \frac{d}{dt}\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u)\|_2^2 - \frac{4K_3^2\mu_1}{l},
\end{aligned} \tag{4.49}$$

and

$$\frac{d}{dt}\left(\frac{l}{2}\|Aw\|_2^2 + 2(f, Aw)\right) = \frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw - \sqrt{\frac{2}{l}}f\|_2^2. \tag{4.50}$$

Therefore, integrating with (4.49)-(4.50), we get from (4.48)

$$\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u) - \sqrt{\frac{2}{l}}f\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)}\right)\\
&+\theta_0\left(\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}h(u) - \sqrt{\frac{2}{l}}f\|_2^2 + \|\Delta\varsigma\|_2^2 + \|\nabla\Delta w\|_2^2 + \|\xi^t\|^2_{\mu,D(A)}\right) \le \tilde{\tilde{C}},
\end{aligned} \tag{4.51}$$

where $\tilde{\tilde{C}} = K_3^2\mu_1^2\left(\frac{1}{\theta l} + \frac{4}{l}\right) + \frac{2\theta_0}{l}\left(K_3^2\mu_1^2 + \|f\|_2^2\right)$. Applying the Gronwell's Lemma, we can easily see that there exists a constant N such that

$$\|Aw\|_2^2 + \|\Delta w_t\|_2^2 + \|\xi^t\|^2_{\mu,D(A)} \le N.$$

Proof of Theorem 4.1

By Lemma 4.1 and Lemma 4.2, we deduce that $(u, u_t, \phi^t) \in \mathcal{H}_1$ and we have

$$\|Au\|_2^2 + \|\Delta u_t\|_2^2 + \|\phi^t\|_{\mu,D(A)} \le N.$$

Now since $u(t,x)$ satisfies (2.6)-(2.8) with initial data (u_0, u_1, ϕ^0), we conclude that

$$\|(u_0, u_1, \phi^0\|_{\mathcal{H}_1} \leq \widehat{N}.$$

Thus $\mathcal{A}$ is a bounded subset of $\mathcal{H}_1$.

References

[1] A.C. Lazer, P.J. McKenna. Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, SIAM Rev. 32 1990; 4: 537-578.

[2] A. Ferrero, F. Gazzola. A partially hinged rectangular plate as a model for suspension bridges, Discrete Contin. Dyn. Syst 2015; 35(12): 5879-5908.

[3] F. Gazzola. Mathematical Models for Suspension Bridges: Nonlinear Structural Instability, Modeling, Simulation and Applications, Springer Verlag 2015; vol.15.

[4] S. Liu, Q. Ma. Global attractors and regularity for the extensible suspension bridge equations with past history, Korean J. Math 2016; 24(3): 375-395.

[5] S. A. Messaoudi, S. E. Mukiawa and C. Enyid. Finite dimensional global attractor for a suspension bridge problem with delay, C. R. Acad. Sci. Paris, Ser.I354 2016; 808-824.

[6] S. A. Messaoudi, A. Bonfoh, S. E. Mukiawa, C. Enyid. The global attractor for a suspension bridge with memory and partially hinged boundary conditions, Z. Angew. Math. Mech 2016; 1-14.

[7] L.D. Humphreys. Numerical mountain pass solutions of a suspension bridge equation, Nonlinear Anal 1997; 28 (11): 1811-1826.

[8] N.U. Ahmed, H. Harbi. Mathematical analysis of dynamic models of suspension bridges, SIAM J. Appl. Math 1998; 58 (3): 853-874.

[9] Q. Ma, C.K. Zhong. Existence of strong solutions and global attractors for the coupled suspension bridge equations, J. Diff. Equations 2009; 246: 3755-3775.

[10] J.Y. Park, J.R. Kang. Pullback $\mathcal{D}$-attractors for non-autonomous suspension bridge equations, Nonlinear Analysis 2009; 71: 4618-4623.

[11] J.Y. Park and J.R. Kang. Global attractors for the suspension bridge equations with nonlinear damping, Quart. Appl. Math 2011; 69: 465-475.

[12] C. M. Dafermos. Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal 1970; 37: 204-229.

[13] R. Temam. Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci, Vol. 68, Springer-Verlag, New York, 1988.

[14] I. Chueshov, I. Lasiecka. Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics, Springer Monographs in Mathematics, Springer, New York, 2010.

Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria

E-mail address: mahdifatimazohra@yahoo.fr

Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria

E-mail address: hakemali@yahoo.com

Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria

E-mail address: ferhat22@hotmail.fr

International Journal of Maps in Mathematics

Volume:1, Issue:2, (2018), Pages:237-248

ISSN: 2636-7467 (Online)

www.journalmim.com

FIXED POINT THEOREMS OF HYBRID PAIRS OF SELF-MAPPINGS IN METRIC SPACE VIA NEW FUNCTIONS

MUZEYYEN SANGURLU SEZEN* AND ARSLAN HOJAT ANSARI

Abstract. In this article, we establish some fixed point theorems for new type generalized contractive mappings involving C-class functions in metric spaces. We provide an example in order to support the useability of our results. These results generalize some well-known results in the literature.

1. Introduction and Preliminaries

Banach [2] introduced a contraction principle which has been extended by many authors to more general contractive conditions in different spaces, for example (see [5–9]) . Kannan obtained the same conclusion as Banach's Theorem with different sufficient conditions (see [11,12]). The conclusion is called Kannan contraction: A mapping T on a metric space (X, d) and if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$\delta(Sx, Ty) \leq \alpha[d(fx, Sx) + d(gy, Ty)]$$

for all $x, y \in X$. Subrahmanyam [17] constructed to show that a metric space having the fixed point property for homeomorphisms need not be metrically topologically complete.

Received:2018-07-03 *Revised:2018-09-06* *Accepted:2018-09-15*

*2010 Mathematics Subject Classification.*47H10,54H25.

*Key words:*Strongly tangential, Hybrid pair, Fixed point

* Corresponding author

Also, he proved that: A metric space (X, d) is complete if and only if every Kannan contraction has a fixed point on X. On the other hand Markin [13] and Nadler [14] initiated the study of fixed points of set valued and multivalued mappings using the Hausdorff metric. Tomar et al. proved strict coincidence and common strict fixed point of strongly tangential hybrid pairs of self-mappings satisfying Kannan type contraction [18].

In this paper, we present new general results of strongly tangential hybrid pairs of self-mappings satisfying Kannan type contraction involving C-class functions. Also we establish coincidence and common fixed point using Hausdorff distance. The obtained results extend many recent results in the literature.

Let (X, d) be a metric space and $CB(X)$ be the family of all nonempty closed and bounded subsets of X. Functions $\delta(A, B)$ and $D(A, B)$ are defined as: $\delta(A, B)$ and $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$

for all $A, B \in CB(X)$. If $A = \{a\}$, then $\delta(A, B) = \delta(a, B)$. If $A = \{a\}$ and $B = \{b\}$, then $\delta(A, B) = d(a, b)$. It follows immediately from the definition of δ that

(a) $\delta(A, B) = \delta(B, A) > 0$,

(b) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$,

(c) $\delta(A, B) = 0$ iff $A = B = \{a\}$,

(d) $\delta(A, A) = diamA$, for all $A, B, C \in CB(X)$.

Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\},$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$ for all $A \in CB(X)$. Also $H(A, B) = 0$ iff $A = B$.

Let (X, d) be a metric space and $h : X \to X$ be a single valued mapping and $T : X \to CB(X)$ be a multivalued mapping. Then (h, T) is called a hybrid pair of mapping. For a multivalued mapping $T : X \to CB(X)$, a point $u \in X$ is

(a) fixed point if $u \in Tu$,

(b) strict fixed point (or a stationary fixed point or absolute fixed point) if $Tu = \{u\}$. For a hybrid pair (h, T), a point $u \in X$ is

(c) coincidence point if $hu = Tu$,

(d) strict coincidence point if $Tu = \{hu\}$,

(e) common fixed point if $u = hu \in Tu$,

(f) common strict fixed point if $hu = Tu = \{u\}$.

Definition 1.1. *[10] Let (X, d) be a metric space. A hybrid pair of mappings (h, T) is weakly commuting if $hTx \in CB(X)$ and $\delta(Thx, hTx) \leq \max\{\delta(hx, Tx), diam(hTx)\}$ for all $x \in X$. Note that if T is a single valued mapping, then the set $\{hTx\}$ consists of a single point. Hence, $diamhTx = 0$ for all $x \in X$ and definition of weak commutativity of a hybrid pair of self mappings reduces to the weak commutativity of a single valued pair of self mappings given by Sessa [15], that is, $d(Thx, hTx) \leq d(hx, Tx)$ for all $x \in X$.*

Definition 1.2. *[16] Let (X, d) be a metric space. A pair of single valued self mappings (h, g) is tangential with respect to a pair of multivalued self mappings (S, T) if*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A \in CB(X)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = z \in A \text{ for some } z \in X.$$

Definition 1.3. *[3] Let (X, d) be a metric space. A pair of single valued self mappings (h, g) is strongly tangential with respect to a pair of multivalued self mappings (S, T) if*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A \in CB(X)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = z \in A \text{ for some } z \in hX \cap gX.$$

Definition 1.4. *[3] Let (X, d) be a metric space. A single valued self mapping h is strongly tangential with respect to multivalued self mapping T if*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n = A \in CB(X)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} hx_n = \lim_{n\to\infty} hy_n = z \in A \text{ for some } z \in hX.$$

Definition 1.5. *[4] Let $h : X \to X$ be a single valued mapping while $T : X \to CB(X)$ be a multivalued mapping. The mapping h is said to be coincidentally idempotent with respect to mapping T, if $hx \in Tx$ imply $hhx = hx$.*

Definition 1.6. *[1] A mapping $f : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following conditions:*

(1) $f(s,t) \leq s$,

(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,\infty)$ and $f(0,0) = 0$.

We denote C-class functions as $\mathcal{C}$.

2. Main Results

Let Φ denote all functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which satisfy

(1) φ is continuous and non-decreasing,

(2) $\varphi(t) = 0$ and only if $t = 0$,

(3) $\varphi(t) < t$, for all $t \in (0,\infty)$ and Ψ denote all functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ which satisfy

(1) $\psi(0) \geq 0$, $\psi(t) > 0$ for all $t > 0$

(2) $\psi(t) \leq t$, for all $t \in (0,\infty)$.

Theorem 2.1. *Let (X,d) be a metric space and $h,g : X \to X$ be single valued and $S,T : X \to CB(X)$ be multi-valued mappings. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(\delta(Sx,Ty)) \leq f(\psi(d(hx,Sx) + d(gy,Ty)), \varphi(d(hx,Sx) + d(gy,Ty))) \tag{2.1}$$

for all $x,y \in X$ and pair of (h,g) is strongly tangential with respect to (S,T). Then pairs (h,S) and (g,T) have strict coincidence point. Morever, h,g,S and T have a unique common strict fixed point if hybrid pairs (h,S) and (g,T) are coincidentally idempotent.

Proof. Suppose that (h,g) is strongly tangential with respect to (S,T). We introduce $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = z \in A = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n$, where $A \in CB(X)$ and $z \in hX \cap gX$. Hence, there exists $u,v \in X$ such that $hu = gv = z$. Now we claim that $z = hu \in Su$. We take $x = u$ and $y = y_n$ in (2.1),

$$\psi(\delta(Su,Ty_n)) \leq f(\psi(d(hu,Su) + d(gy_n,Ty_n)), \varphi(d(hu,Su) + d(gy_n,Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu,Su)) \leq \psi(\delta(Su,A)) \leq f(\psi(d(hu,Su)), \varphi(d(hu,Su))) \leq \psi(d(hu,Su)).$$

Then, we have

$$f(\psi(d(hu,Su)), \varphi(d(hu,Su))) = \psi(d(hu,Su)).$$

From the property of f, we get

$$\psi(d(hu,Su))=0 \text{ or } \varphi(d(hu,Su))=0 \text{ so } d(hu,Su)=0.$$

Therefore, $hu\in Su$, that is, $\delta(Su,A)=0$ and so we get $Su=\{hu\}$. Hence, h and S have a strict coincidence point. Now we show that $z=gv\in Tv$, using $x=x_n$ and $y=v$ in (2.1)

$$\psi(\delta(Sx_n,Tv))\leq f(\psi(d(hx_n,Sx_n)+d(gv,Tv)),\varphi(d(hx_n,Sx_n)+d(gv,Tv)).$$

Taking limit as $n\to\infty$, from property of f, we get

$$\psi(\delta(A,Tv))\leq f(\psi(d(gv,Tv)),\varphi(d(gv,Tv))\leq\psi(d(gv,Tv)))\leq\psi(d(z,A)).$$

From $gv=z\in A$, we have

$$\psi(d(gv,Tv))\leq\psi(\delta(A,Tv))\leq f(\psi(d(gv,Tv)),\varphi(d(gv,Tv)))\leq\psi(d(gv,Tv)).$$

So,

$$\psi(d(gv,Tv))=0 \text{ or } \varphi(d(gv,Tv))=0 \text{ so } d(gv,Tv)=0.$$

Therefore, $gv\in Tv$, that is, $\delta(Tu,A)=0$ and so we have $Tv=\{gu\}$. Hence, g and T have a strict coincidence point. Thus, $z\in Su=Tv=\{z\}$. Now since (h,S) is coincidentally idempotent, $hu\in Su$ implies $hhu=hu\in Su$. Now we claim that $z=hz\in Sz$. We take $x=z$ and $y=y_n$ in (2.1),

$$\psi(\delta(Sz,Ty_n))\leq f(\psi(d(hu,Su)+d(gy_n,Ty_n)),\varphi(d(hu,Su)+d(gy_n,Ty_n))).$$

Taking limit as $n\to\infty$, we get,

$$\psi(d(hu,Su))\leq\psi(\delta(Sz,A))\leq f(\psi(d(hz,Sz)),\varphi(d(hz,Sz)))\leq\psi(d(hz,Sz)).$$

Since $hz=z\in A$, we get,

$$\psi(d(hz,Sz))\leq\psi(\delta(Sz,A))\leq f(\psi(d(hz,Sz)),\varphi(d(hz,Sz)))\leq\psi(d(hz,Sz)).$$

So,

$$\psi(d(hz,Sz))=0 \text{ or } \varphi(d(hz,Sz))=0 \text{ so } d(hz,Sz)=0.$$

Therefore, $hz\in Sz$, that is, $\delta(Sz,A)=0$ and so we have $Sz=\{hz=z\}$. Similarly, (g,T) is coincidentally idempotent $gv\in Tv$ implies $ggv=gv\in Tv$.

Now we claim that $z=gz\in Tz$. We take $x=x_n$ and $y=z$ in (2.1),

$$\psi(\delta(Sx_n,Tz))\leq f(\psi(d(hx_n,Sx_n)+d(gz,Tz)),\varphi(d(hx_n,Sx_n)+d(gz,Tz))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(\delta(A, Tz)) \leq f(\psi(d(z, A) + d(gz, Tz)), \varphi(d(z, A) + d(gz, Tz)))$$

or

$$\psi(\delta(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Since $gz = z \in A$, we get,

$$\psi(d(gv, Tz)) \leq \psi(\delta(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Then,

$$\psi(d(gz, Tz)) = 0 \text{ or } \varphi(d(gz, Tz)) = 0 \text{ so } d(gz, Tz) = 0.$$

Hence, $gz \in Tz$, that is, $\delta(Tz, A) = 0$ and so we get $Tz = \{gz = z\}$. Therefore z is a common strict fixed point of h, g, T and S.

Let z and w be two common strict fixed points such that $z \neq w$. Now from (2.1), we have,

$$\begin{aligned} \psi(\delta(Sz, Tw)) &\leq f(\psi(d(hz, Sz) + d(gw, Tw)), \varphi(d(hz, Sz) + d(gw, Tw))) \\ &\leq f(0, 0) \leq 0. \end{aligned}$$

$$\delta(Sz, Tw) \leq 0$$

but

$$\delta(Sz, Tw) > 0$$

which is a contradiction. Hence, z is a unique common strict fixed point of h, g, T and S.

Taking $h = g$ and $T = S$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let (X, d) be a metric space and $h : X \to X$ be single valued and $T : X \to CB(X)$ be multi-valued mapping. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(\delta(Tx, Ty)) \leq f(\psi(d(hx, Tx) + d(hy, Ty)), \varphi(d(hx, Tx) + d(hy, Ty))) \tag{2.2}$$

for all $x, y \in X$ and pair of h is strongly tangential with respect to T. Then h and T have strict coincidence point. Morever, h and T have a unique common strict fixed point if hybrid pair (h, T) is coincidentally idempotent.

Theorem 2.2. *Let (X,d) be a metric space and $h,g : X \to X$ be single valued and $S,T : X \to CB(X)$ be multi-valued mapping. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(H(Sx,Ty)) \leq f(\psi(d(hx,Sx)+d(gy,Ty)), \varphi(d(hx,Sx)+d(gy,Ty)) \tag{2.3}$$

for all $x,y \in X$ and pair of (h,g) is strongly tangential with respect to (S,T). Then pairs (h,S) and (g,T) have coincidence point. Morever, h,g,S and T have a common fixed point if hybrid pairs (h,S) and (g,T) are coincidentally idempotent.

Proof. Suppose that (h,g) is strongly tangential with respect to (S,T). We introduce $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = z \in A = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n$, where $A \in CB(X)$ and $z \in hX \cap gX$. Hence, there exists $u,v \in X$ such that $fu = gv = z$. Now we claim that $z = hu \in Su$. We take $x = u$ and $y = y_n$ in (2.3),

$$\psi(H(Su,Ty_n)) \leq f(\psi(d(hu,Su)+d(gy_n,Ty_n)), \varphi(d(hu,Su)+d(gy_n,Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu,Su)) \leq \psi(H(Su,A)) \leq f(\psi(d(hu,Su)), \varphi(d(hu,Su))) \leq \psi(d(hu,Su)).$$

Then, we have

$$f(\psi(d(hu,Su)), \varphi(d(hu,Su))) = \psi(d(hu,Su)).$$

From the property of f, we get

$$\psi(d(hu,Su)) = 0 \text{ or } \varphi(d(hu,Su)) = 0 \text{ so } d(hu,Su) = 0.$$

Hence, $hu \in Su$. Hence, h and S have a coincidence point. Now we show that $z = gv \in Tv$, using $x = x_n$ and $y = v$ in (2.3)

$$\psi(H(Sx_n,Tv)) \leq f(\psi(d(hx_n,Sx_n)+d(gv,Tv)), \varphi(d(hx_n,Sx_n)+d(gv,Tv))).$$

Taking limit as $n \to \infty$, from property of f, we get

$$\psi(H(A,Tv)) \leq f(\psi(d(gv,Tv)), \varphi(d(gv,Tv))) \leq \psi(d(gv,Tv)) \leq \psi(d(z,A)).$$

From $gv = z \in A$, we have

$$\psi(d(gv,Tv)) \leq \psi(H(A,Tv)) \leq f(\psi(d(gv,Tv)), \varphi(d(gv,Tv))) \leq \psi(d(gv,Tv)).$$

So,

$$\psi(d(gv,Tv)) = 0 \text{ or } \varphi(d(gv,Tv)) = 0 \text{ so } d(gv,Tv) = 0.$$

Therefore, $gv \in Tv$. Hence, g and T have a coincidence point. Thus, $z \in Su = Tv = \{z\}$. Now since (h, S) is coincidentally idempotent, $hu \in Su$ implies $hhu = hu \in Su$. Now we claim that $z = hz \in Sz$. We take $x = z$ and $y = y_n$ in (2.3),

$$\psi(H(Sz, Ty_n)) \leq f(\psi(d(hu, Su) + d(gy_n, Ty_n)), \varphi(d(fu, Su) + d(gy_n, Ty_n))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(d(hu, Su)) \leq \psi(H(Sz, A)) \leq f(\psi(d(hz, Sz)), \varphi(d(hz, Sz))) \leq \psi(d(hz, Sz)).$$

Since $hz = z \in A$, we get,

$$\psi(d(hz, Sz)) \leq \psi(H(Sz, A)) \leq f(\psi(d(hz, Sz)), \varphi(d(hz, Sz))) \leq \psi(d(hz, Sz)).$$

So,

$$\psi(d(hz, Sz)) = 0 \text{ or } \varphi(d(hz, Sz)) = 0 \text{ so } d(fz, Sz) = 0.$$

Hence, $hz \in Sz$. Similarly, (g, T) is coincidentally idempotent $gv \in Tv$ implies $ggv = gv \in Tv$.

Now we claim that $z = gz \in Tz$. We take $x = x_n$ and $y = z$ in (2.3),

$$\psi(H(Sx_n, Tz)) \leq f(\psi(d(hx_n, Sx_n) + d(gz, Tz)), \varphi(d(hx_n, Sx_n) + d(gz, Tz))).$$

Taking limit as $n \to \infty$, we get,

$$\psi(H(A, Tz)) \leq f(\psi(d(z, A) + d(gz, Tz)), \varphi(d(z, A) + d(gz, Tz)))$$

or

$$\psi(H(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Since $gz = z \in A$, we get,

$$\psi(d(gv, Tz)) \leq \psi(H(A, Tz)) \leq f(\psi(d(gz, Tz)), \varphi(d(gz, Tz))).$$

Then,

$$\psi(d(gz, Tz)) = 0 \text{ or } \varphi(d(gz, Tz)) = 0 \text{ so } d(gz, Tz) = 0.$$

Thus, $gz \in Tz$. Therefore z is a common fixed point of h, g, T and S.

Let z and w be two common fixed points such that $z \neq w$. Now from condition (1), we have,

$$\begin{aligned}\psi(H(Sz, Tw)) &\leq f(\psi(d(hz, Sz) + d(gw, Tw)), \varphi(d(hz, Sz) + d(gw, Tw))) \\ &\leq f(0, 0) \leq 0.\end{aligned}$$

$$H(Sz, Tw) \leq 0$$

but

$$H(Sz, Tw) > 0$$

which is a contradiction. Hence, z is a unique common fixed point of h, g, T and S.

Taking $h = g$ and $T = S$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.2. *Let (X, d) be a metric space and $h : X \to X$ be single valued and $T : X \to CB(X)$ be multi-valued mapping. If $\varphi \in \Phi$, $\psi \in \Psi$ and f is element of $\mathcal{C}$ such that*

$$\psi(H(Tx, Ty)) \leq f(\psi(d(hx, Tx) + d(hy, Ty)), \varphi(d(hx, Tx) + d(hy, Ty)) \tag{2.4}$$

for all $x, y \in X$ and pair of h is strongly tangential with respect to T. Then h and T have coincidence point. Morever, h and T have a common fixed point if hybrid pair (h, T) is coincidentally idempotent.

Example 2.1. *Let $X = [0, 5]$, d be usual metric on X, Let a hybrid pair of mappings h,T : $X \to X$ by*

$$hx = \begin{cases} \frac{4-x}{2} & , \quad x \in [0,2] \\ 4 & , \quad x \in (2,5] \end{cases} \quad and \quad Tx = \begin{cases} \{\frac{4}{3}\} & , \quad x \in [0,2] \\ 1 & , \quad x \in (2,5] \end{cases}.$$

Define $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{t}{5}, \varphi(t) = 3t$ and $F(s, t) = ks$ for $k \in (0, 1)$. Consider two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n = \frac{4}{3} - \frac{1}{n}$ and $y_n = \frac{4}{3}$ for all $n > 1$. Clearly $\lim\limits_{n\to\infty} hx_n = \lim\limits_{n\to\infty} hy_n = \frac{4}{3} \in \{\frac{4}{3}\} = \lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Ty_n$ and $\frac{4}{3} \in hX$. Hence h is strongly tangential with respect to T. The point $z = \frac{4}{3}$ is a strict coincidence point and $hh\frac{4}{3} = h\frac{4}{3}$, that is, (h, T) is coincidentally idempotent.
For $x, y \in [0, 2]$, we have

$$\psi(\delta(Tx, Ty)) = 0 \leq k.\frac{2d(hx, Tx)}{5},$$

for $x \in [0, 2]$ and $y \in (2, 5]$, we have

$$\psi(\delta(Tx, Ty)) = \frac{1}{3} \leq k.\frac{d(hx, Tx) + d(hy, Ty)}{5},$$

for $x, y \in (2, 5]$, we have

$$\psi(\delta(Tx, Ty)) = 0 \leq k.\frac{2d(hx, Tx)}{5},$$

for $x \in (2, 5]$ and $y \in [0, 2]$, we have

$$\psi(\delta(Tx, Ty)) = \frac{1}{3} \leq k.\frac{d(hx, Tx) + d(hy, Ty)}{5}.$$

Thus h and T satisfy Corollary 2.1, for $k = \frac{1}{3} \in (0, 1)$. Also $T\frac{4}{3} = \{h\frac{4}{3}\} = \{\frac{4}{3}\}$, that is, $\frac{4}{3}$ is the unique common strict fixed point of h and T.

3. Application

In this section, we generalize the results of Theorem 4.1 given by Tomar et al. [18].

Let $B(W)$ be the set of all closed and bounded real-valued functions on W. For an arbitrary $p, k \in B(W)$ define $\|p\| = \sup_{x\in W} |p(x)|$, $\|k\| = \sup_{x\in W} |k(x)|$ and $\delta(p,k) = \sup_{x\in W} |p(x) - k(x)|$. Also, $(B(W), \|.\|)$ is a Banach space wherein convergence is uniform. Consider the operators $T_i, A_i : B(W) \to B(W)$ given by

$$\begin{cases} T_i p(x) = \sup_{y\in D}\{g(x,y) + G_i(x,y,p(\tau(x,y)))\}, & i = 1,2, \\ A_i k(x) = \sup_{y\in W}\{g^{'}(x,y) + G_i^{'}(x,y,p(\tau(x,y)))\}, & i = 1,2, \end{cases} \tag{3.1}$$

for $p, k \in B(W)$, where $\tau : W \times D \to W$, $g.g^{'} : W \times D \to \mathbb{R}$, $G_i.G_i^{'} : W \times D \times \mathbb{R} \to \mathbb{R}$ are given mappings, while $W \in U$ is a state space, $D \in V$ is a decision space and U, V are Banach spaces. These mappings are well-defined if the functions $g_i, g_i^{'}$,G_i and $G_i^{'}$ are bounded. Also, denote

$$\Theta(p,k) = f(\psi(d(A_1p, T_1p) + d(A_2k, T_2k)), \varphi(d(A_1p, T_1p) + d(A_2k, T_2k)))$$

for $p, k \in B(W)$.

Theorem 3.1. *Let $T_i, A_i : B(W) \to B(W)$ given by (3.1), for $i = 1,2$. Suppose that the following conditions hold:*

(1) For all $x \in W, y \in D$, $|G_1(x,y,p(\tau(x,y)) - G_2(x,y,p(\tau(x,y))| \le \Theta(p,k)$,

(2) For $i = 1,2$, $g.g^{'} : W \times D \to \mathbb{R}$, $G_i.G_i^{'} : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions,

(3) There exists sequences $\{p_n\}$,$\{k_n\} \in B(W)$ and functions $p^* \in B(W)$ such that

$$\lim_{n\to\infty} T_1 p_n = \lim_{n\to\infty} T_2 k_n = A \in B(W)$$

and

$$\lim_{n\to\infty} A_1 p_n = \lim_{n\to\infty} A_2 k_n = p^* \in A \text{ and } p^* \in A_1 \cap A_2,$$

(4) $A_1A_1p = A_1p$ whenever $A_1p \in T_1p$ and $A_2A_2k = A_2k$ whenever $A_2k \in T_2k$ for some $p, k \in B(W)$. Then, the equation system (5) has a bounded solution.

Proof. Let $\delta(h,k) = \sup_{x\in W} |h(x) - k(x)|$ for any $h, k \in B(W)$ and $\psi(t) = t$ for $t \in [0, +\infty)$. Let λ be an arbitrary positive number, $x \in W$. Then there exists $y_1, y_2 \in D$ such that

$$T_1 h(x) < g(x,y_1) + G_1(x,y_1,h(\tau(x,y_1)) + \lambda \tag{3.2}$$

$$T_2k(x) < g(x,y_2) + G_2(x,y_2,k(\tau(x,y_2)) + \lambda. \tag{3.3}$$

From the definition, we have

$$T_1h(x) > g(x,y_2) + G_1(x,y_2,h(\tau(x,y_2)) + \lambda \tag{3.4}$$

$$T_2k(x) > g(x,y_1) + G_2(x,y_1,k(\tau(x,y_1)) + \lambda. \tag{3.5}$$

From (3.2) and (3.5), we get

$$T_2k(x) - T_1h(x) < \Theta(h,k) + \lambda. \tag{3.6}$$

Combining, we get

$$|T_1h(x) - T_2k(x)| < \Theta(h,k) + \lambda.$$

Implying thereby

$$\delta(T_1h(x), T_2k(x)) < \Theta(h,k) + \lambda. \tag{3.7}$$

Also, (3.7) does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily. Hence, we obtain

$$\delta(T_1h(x), T_2k(x)) < \Theta(h,k)$$

for each $t \in (0,\infty)$. From condition (3), (A_1, A_2) is strongly tangential with respect to (T_1, T_2). Thus, from condition (4) and taking $h = A_1, S = T_1, g = A_2, T = T_2$ all the conditions of Theorem 2.1 are satisfied. Hence, from Theorem 2.1, T_1,T_2, A_1 and A_2 have a unique common fixed point, the system of functional equations (3.1) has a unique bounded solution.

References

[1] Ansari AH. Note on $\psi-\phi$-contractive type mappings and related fixed point, The 2nd Regional Conference on Mathematics And Applications, Payame Noor University 2014; 377-380.

[2] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. 1922; 3; 133-181.

[3] Chauhan S, Imdad M, Karapinar E, Fisher B. An integral type fixed point theorem for multi-valued mappings employing strongly tangential property, J. Egyptian Math. Soc 2014; 22(2); 258-264.

[4] Imdad M, Ahmad A, Kumar S. On nonlinear nonself hybrid contractions, Radovi Matematicki 2001; 10; 233-244.

[5] Isik H, Turkoglu D. Fixed point theorems for weakly contractive mappings in partially ordered metric-like spaces, Fixed Point Theory Appl. 2013; 2013(51); 12 pages.

[6] Isik H, Turkoglu D. Coupled fixed point theorems for new contractive mixed monotone mappings and applications to integral equations, Filomat 2014; 28(6); 1253-1264.

[7] Isik H, Samet B, Vetro C. Cyclic admissible contraction and applications to functional equations in dynamic programming, Fixed Point Theory Appl. 2015; 2015(163); DOI 10.1186/s13663-015-0410-6.

[8] Isik H, Turkoglu D. Generalized weakly α-contractive mappings and applications to ordinary differential equations. Miskolc Mathematical Notes 2016; 17(1); 365-379.

[9] Isik H, Radenovic S. A new version of coupled fixed point results in ordered metric spaces with applications. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics 2017; 79(2), 131-138.

[10] Kaneko H. A common fixed point of weakly commuting multivalued mappings, Math. Japon. 1988; 33(5); 741-744.

[11] Kannan R. Some results on fixed points, Bull. Calcutta Math. Soc 1968; 60; 71-76.

[12] Kannan R. Some results on fixed points. II, American Mathematical Monthly 1969; 76(4); 405-408.

[13] Markin J. A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc 1968; 74; 639-640.

[14] Nadler SB. Multi-valued contraction mappings, Pacific J. Math 1969; 30(2); 475-486.

[15] Sessa S. On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math 1982; 32(46); 149-153.

[16] Sintunavarat W, Kumam P. Gregus-type common fixed point theorems for tangential multi-valued mappings of integral type in metric spaces, Int. J. Math. Math. Sci 2011; 12 pages, DOI:10.1155/2011/923458.

[17] Subrahmanyam V. Completeness and fixed-points, Monatsh. Math 1975; 80(4); 325-330.

[18] Tomar A, Upadhyay S, Sharma R. Strict Coincidence and Common Strict Fixed Point of Hybrid Pairs of Self-mappings with Application, Math. Sci. and Appl. E-Notes 2017; 5(2); 51-59.

Giresun University, Department of Mathematics, 28100, Giresun-Turkey

E-mail address: muzeyyen.sezen@giresun.edu.tr

Department of Mathematics, Faculty of Science, Iran

E-mail address: mathanalsisamir4@gmail.com

International Journal of Maps in Mathematics
Volume:1, Issue:2, (2018), Pages:249-257
ISSN: 2636-7467 (Online)
www.journalmim.com

EXTENDED TRAVELING WAVE SOLUTIONS FOR SOME INTEGRO PARTIAL DIFFERENTIAL EQUATIONS

SERIFE MUGE EGE

Abstract. In this study, an extended method is implemented to find traveling wave solutions of two integro partial differential equations. The exact particular solutions containing hyperbolic function type are obtained. By using symbolic computation it is shown that this method is efficient mathematical tool for solving problems in nonlinear science.

1. Introduction

Many nonlinear physical pheonemena such as liquid dynamics, elasticity, chemical kinematics, relativity, optical fiber etc. are modelled by nonlinear partial differential equations. Therefore traveling wave solutions of nonlinear partial differential equations have importance in real world problems. Due to these solutions give information about the character of physical events, it is required to powerful methods such as the auxiliary equation method [1], extended auxiliary equation method [2], Painleve method [3], inverse scattering method [4], simple equation method [5], modified simple equation method [6, 7, 8, 9], extended simple equation method [10], G'/G expansion method [11, 12, 13], $\tan(\phi(\xi)/2)-$ expansion method [14], tanh method [15], extended tanh method [16], $\exp(-phi(xi))$-expansion method [17], subequation method [18],modified Kudryashov method [19], generalized Kudryashov method

Received:2018-07-13 *Revised:2018-07-30* *Accepted:2018-08-29*

*2010 Mathematics Subject Classification.*35Q79; 35Q35; 35Q40.

Key words: Extended method, integro partial differential equations, $(1+1)$ dimensional Ito equation, $(2+1)$ dimensional Ito equation.

[20, 21, 26], extended Kudryashov method [22], ansatz method [23] and so on.

In this paper, by inspiring the modified Kudryashov method, an extended method is executed to find the traveling wave solutions for two integro partial differential equations, namely, $(1+1)$ - dimensional and $(2+1)$ - dimensional Ito's equations given as [24, 25, 26]:

$$v_{tt} + v_{xxxt} + 3(v_x u_t + v v_{xt}) + 3v_{xx} \int_{-\infty}^{x} v_t dx = 0$$

where v is the function of (x,t) and

$$v_{tt} + v_{xxxt} + 3(2v_x v_t + v v_{xt}) + 3v_{xx} \int_{-\infty}^{x} v_t dx + \alpha v_{yt} + \beta v_{xt} = 0 \tag{1.1}$$

where v is the function of (x,y,t).

The remnant of this paper organized as follows: In the following section we have a brief review on the extended method. In section 3, we use this method to get traveling wave solutions of Ito's equations. Finally, conclusions are given in Section 4.

2. Methodology

The extended method is described systematically in this section [22].

Step 1. We suppose that given nonlinear partial differential equation for $u(x,t)$ to be in the form:

$$P\left(u, u_t, u_x, u_y, u_z, u_{xy}, u_{yz}, u_{xz}, ...\right) = 0 \tag{2.2}$$

which can be reduced to an ordinary differential equation. Then Eq.(2.2) reduces to a nonlinear ordinary differential equation of the form:

$$H(u, u_\mu, u_{\mu\mu}, u_{\mu\mu\mu}, \ldots) = 0 \tag{2.3}$$

under the wave transformation

$$u(x,y,z,\ldots,t) = u(\mu), \quad \mu = k(x+ct) \quad \texttt{or} \quad \mu = x - ct, \tag{2.4}$$

where k and c are constants.

Step 2. Suppose that the traveling wave solutions of Eq.(2.3) to be as follows:

$$u(\mu) = \sum_{i=0}^{N} a_i Z^i(\mu) \tag{2.5}$$

where $a_i(i = 0,1,2,..,N)$ are constants such that $a_N \neq 0$ and $Z = \pm\frac{1}{\sqrt{1 \pm a^{2\mu}}}$. The function Z is the solution of equation of the auxilary ordinary differential equation

$$Z_\mu = lna(Z^3 - Z). \tag{2.6}$$

Step 3. In order to calculate the positive integer N in formula (2.5) we consider the homogenous balance between the highest order nonlinear terms and highest order derivatives in Eq. (2.3). Supposing $u^s(\mu)u^{(l)}(\mu)$ and $(u^{(r)}(\mu))^p$ are the highest order nonlinear terms of Eq. (2.3) and we have

$$N = \frac{2(l-pr)}{p-s-1}. \tag{2.7}$$

Step 4. Substituting Eq.(2.5) into Eq.(2.3) and equating the coefficients of Z^i to zero, we obtain a system of algebraic equations. By solving this system with the help of Mathematica packet program, we get the traveling wave solutions of Eq.(2.3).

3. Applications

3.1. **(1+1) dimensional integro-differential Ito Equation.** We first apply the method to $(1+1)$ - dimensional integro-differential Ito equation in the form:

$$v_{tt} + v_{xxxt} + 3(v_x u_t + v v_{xt}) + 3v_{xx}\int_{-\infty}^{x} v_t dx = 0 \tag{3.8}$$

where v is the function of (x,t).

We use the transformation

$$v(x,t) = u_x(x,t).$$

This transformation carries Eq.(3.8) into following differential equation:

$$u_{xtt} + u_{xxxxt} + 3(u_{xx}u_{xt} + u_x u_{xxt}) + 3u_{xxx}u_t = 0. \tag{3.9}$$

Then, using travelig wave transformation (2.4) we have

$$-cu''' + u^{(v)} - 3c(u''u'' + u'u'') - 3cu'''u' = 0. \tag{3.10}$$

where $' = \frac{d}{d\xi}$. By integrating Eq.(3.10), we obtain, upon setting the integration constant to zero,

$$u''' + 3c(u')^2 - cu' = 0. \tag{3.11}$$

Then using the transformation $\omega = u'$ Eq.(3.11) can be written as

$$\omega'' + 3\omega^2 - c\omega = 0. \tag{3.12}$$

Also we take

$$\omega(\mu) = \sum_{i=0}^{N} a_i Z^i \tag{3.13}$$

where $Z(\mu) = \pm\frac{1}{(1\pm e^{2\mu})^{1/2}}$. We note that the function Z is the solution of $Z_\mu = Z^3 - Z$. Balancing the highest order derivative and nonlinear term we calculate

$$N = 4. \tag{3.14}$$

Thus, we have

$$\omega(\mu) = a_0 + a_1 Z(\mu) + a_2 Z^2(\mu) + a_3 Z^3(\mu) + a_4 Z^4(\mu) \tag{3.15}$$

and substituting derivatives of $\omega(\mu)$ with respect to μ in Eq.(3.15). The required derivatives in Eq. (3.12) are obtained

$$\omega_\mu = (Z^3 - Z)(a_1 + 2a_2 Z + 3a_3 Z^2 + 4a_4 Z^3), \tag{3.16}$$

$$\omega_{\mu\mu} = (Z^3 - Z)[24a_4 Z^5 + 15a_3 Z^4 + (8a_2 - 16a_4)Z^3 \tag{3.17}$$

$$+ (3a_1 - 9a_3)Z^2 - 4a_2 - a_1]. \tag{3.18}$$

Substituting derivatives Eq.(3.15) and Eq.(3.16) into Eq.(3.12) and accumulate the coefficient of each power of Z^i, setting each of coefficient to zero, solving the resulting system of algebraic equations we get the following solutions:

Case 1:

$$a_0 = -\frac{4}{3}, \quad a_1 = a_1, \quad a_2 = 8, \quad a_3 = a_3, \tag{3.19}$$

$$a_4 = -8, \quad c = -4. \tag{3.20}$$

Inserting Eq.(3.19) into Eq.(3.15), we obtain the following solutions of Eq.(3.8) with respect to traveling wave transformation (2.4)

$$v_1(\mu) = -\frac{4}{3} + \frac{a_1(1+e^{2\mu}) + a_3}{(1+e^{2\mu})^{3/2}} + \frac{2}{cosh^2(\mu)}, \tag{3.21}$$

$$v_2(\mu) = -\frac{4}{3} + \frac{a_1(1-e^{2\mu}) + a_3}{(1-e^{2\mu})^{3/2}} - \frac{2}{sinh^2(\mu)}. \tag{3.22}$$

Thus, we obtain new exact solutions to Eq.(3.8)

$$v_1(x,t) = -\frac{4}{3} + \frac{a_1(1+e^{2x+8t}) + a_3}{(1+e^{2x+8t})^{3/2}} + \frac{2}{cosh^2(x+4t)}, \tag{3.23}$$

$$v_2(x,t) = -\frac{4}{3} + \frac{a_1(1-e^{2x+8t}) + a_3}{(1-e^{2x+8t})^{3/2}} - \frac{2}{sinh^2(x+4t)}. \tag{3.24}$$

Case 2:

$$a_0 = 0, \quad a_1 = a_1, \quad a_2 = -8, \quad a_3 = a_3, \tag{3.25}$$

$$a_4 = -8, \quad c = 4 \tag{3.26}$$

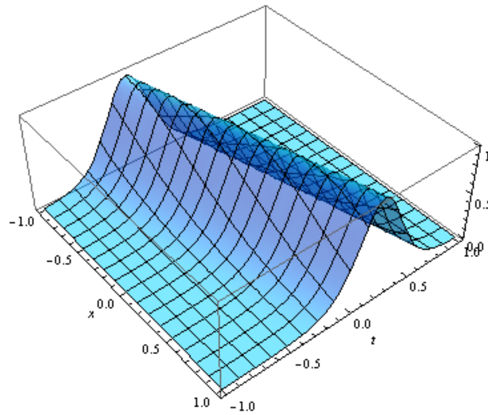

FIGURE 1. The exact solution v_1 of Eq.(3.8)

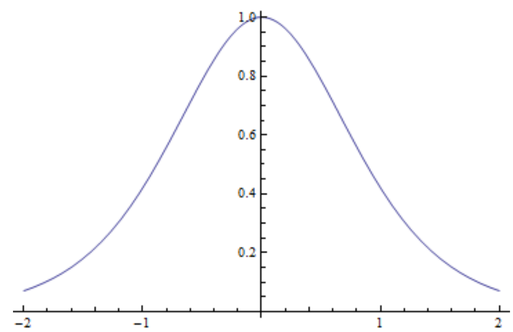

FIGURE 2. The projection of v_1 at $t = 0$

Inserting Eq.(3.25) into Eq.(3.15), we obtain the following solutions of Eq.(3.8) with respect to traveling wave transformation (2.4)

$$v_3(\mu) = \frac{a_1(1+e^{2\mu}) + a_3}{(1+e^{2\mu})^{3/2}} + \frac{2}{cosh^2(\mu)}, \tag{3.27}$$

$$v_4(\mu) = \frac{a_1(1-e^{2\mu}) + a_3}{(1-e^{2\mu})^{3/2}} - \frac{2}{sinh^2(\mu)}. \tag{3.28}$$

Thus, we get new exact solutions to Eq.(3.8)

$$u_3(x,t) = \frac{a_1(1+e^{2x-8t}) + a_3}{(1+e^{2x-8t})^{3/2}} + \frac{2}{cosh^2(x-4t)}, \tag{3.29}$$

$$u_4(x,t) = \frac{a_1(1-e^{2x-8t}) + a_3}{(1-e^{2x-8t})^{3/2}} - \frac{2}{sinh^2(x-4t)}. \tag{3.30}$$

3.2. **(2+1) dimensional integro-differential Ito Equation.** We secondly apply the method to $(2+1)$ - dimensional integro-differential Ito equation in the form:

$$v_{tt} + v_{xxxt} + 3(2v_x v_t + v v_{xt}) + 3v_{xx}\int_{-\infty}^{x} v_t dx + \alpha v_{yt} + \beta v_{xt} = 0 \tag{3.31}$$

where v is the function of (x, y, t).

Using the transformation

$$v(x,t) = u_x(x,t)$$

Eq.(3.31) turns into following differential equation:

$$u_{xtt} + u_{xxxxt} + 3(2u_{xx}u_{xt} + u_x u_{xxt}) + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt} = 0. \tag{3.32}$$

By considering the traveling wave transformation $\mu = x + y - ct$, Eq.(3.33) can be reduced to the following ordinary differential equation:

$$(c - \alpha - \beta)u''' - u^{(v)} - 3((u')^2)'' = 0 \tag{3.33}$$

where $' = \frac{d}{d\xi}$. If we integrate twice, we get

$$(c - \alpha - \beta)u' - u''' - 3(v')^2 = 0. \tag{3.34}$$

Then using the transformation $\omega = u'$ Eq.(3.34) can be written as

$$(c - \alpha - \beta)\omega - \omega'' - 3\omega^2 = 0. \tag{3.35}$$

Also we take

$$\omega(\mu) = \sum_{i=0}^{N} a_i Z^i \tag{3.36}$$

where $Z(\mu) = \pm\frac{1}{(1\pm e^{2\mu})^{1/2}}$. We note that the function Z is the solution of $Z_\mu = Z^3 - Z$. Balancing the highest order derivative and nonlinear term we calculate

$$N = 4. \tag{3.37}$$

Thus, we have

$$\omega(\mu) = a_0 + a_1 Z(\mu) + a_2 Z^2(\mu) + a_3 Z^3(\mu) + a_4 Z^4(\mu) \tag{3.38}$$

and substituting derivatives of $\omega(\mu)$ with respect to μ in Eq.(3.38). The required derivatives in Eq. (3.35) are obtained

$$\omega_\mu = (Z^3 - Z)(a_1 + 2a_2 Z + 3a_3 Z^2 + 4a_4 Z^3), \tag{3.39}$$

$$\begin{aligned} \omega_{\mu\mu} &= (Z^3 - Z)[24a_4 Z^5 + 15a_3 Z^4 + (8a_2 - 16a_4)Z^3 \\ &+ (3a_1 - 9a_3)Z^2 - 4a_2 - a_1]. \end{aligned} \tag{3.40}$$

Substituting derivatives Eq.(3.38) and Eq.(3.39) into Eq.(3.35) and accumulating the coefficient of each power of Z^i, setting each of coefficient to zero, solving the resulting algebraic

equation system we obtain the following solutions:

Case 1:

$$a_0 = -\frac{4}{3}, \quad a_1 = 0, \quad a_2 = 8, \quad a_3 = 0, \tag{3.41}$$

$$a_4 = -8, \quad c = -4 + \alpha + \beta \tag{3.42}$$

Inserting Eq.(3.41) into Eq.(3.38), we obtain the following solutions of Eq.(3.31) with respect to traveling wave transformation $\mu = x + y - ct$

$$v_1(\mu) = -\frac{4}{3} + \frac{2}{cosh^2(\mu)}, \tag{3.43}$$

$$v_2(\mu) = -\frac{4}{3} - \frac{2}{sinh^2(\mu)}. \tag{3.44}$$

Thus, we obtain new exact solutions to Eq.(3.31) in the form:

$$v_1(x, y, t) = -\frac{4}{3} + \frac{2}{cosh^2(x + y - (4 - \alpha - \beta)t)}, \tag{3.45}$$

$$v_2(x, t) = -\frac{4}{3} - \frac{2}{sinh^2(x + y - (4 - \alpha - \beta)t)}. \tag{3.46}$$

Case 2:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 8, \quad a_3 = 0, \tag{3.47}$$

$$a_4 = -8, \quad c = 4 + \alpha + \beta \tag{3.48}$$

Inserting Eq.(3.47) into Eq.(3.38), we get the following solutions of Eq(3.31) with respect to traveling wave transformation $\mu = x + y - ct$:

$$v_1(\mu) = \frac{2}{cosh^2(\mu)}, \tag{3.49}$$

$$v_2(\mu) = \frac{2}{sinh^2(\mu)}. \tag{3.50}$$

Thus, we obtain new exact solutions to Eq.(3.31) in the form:

$$v_3(x, t) = \frac{2}{cosh^2(x + y - (4 + \alpha + \beta)t)}, \tag{3.51}$$

$$v_4(x, t) = \frac{2}{sinh^2(x + y - (4 + \alpha + \beta)t)}. \tag{3.52}$$

4. Conclusion

In this work, the extended method is executed to construct exact solutions of nonlinear integro partial differential equations with constant coefficients. By using the proposed method we have successfully obtained analytical solutions of $(1+1)$ - dimensional and $(2+1)$ - dimensional Ito equations. Besides the solutions in [26] , hyperbolic function type solutions are obtained. In addition, change in the parameters effects the speed of the wave. The obtained solutions may have importance for some special technological and physical events. It can be concluded that this method is standard, effective and also convenient for solving nonlinear integro partial differential equations.

References

[1] Tariqab Aly KU, Seadawy R. On the soliton solutions to the modified Benjamin-Bona-Mahony and coupled Drinfel'd-Sokolov-Wilson models and its applications, Journal of King Saud University - Science 2018; doi.org/10.1016/j.jksus.2018.03.01.

[2] Tariqab Aly KU, Seadawy R. The extended auxiliary equation method for the KdV equation with variable coefficients, Chinese Physics B 2011; 20: 1-5.

[3] Xu GQ ,Deng SF. Painlev analysis, integrability and exact solutions for a $(2+1)$-dimensional generalized Nizhnik-Novikov-Veselov equation, The European Physical Journal Plus 2016; 131:385.

[4] Ablowitz MJ, Segur H. Solitons and the Inverse Scattering Transform, Studies in Applied and Numerical Mathematics 1981.

[5] Nofal TA. Simple equation method for nonlinear partial differential equations and its applications, Journal of the Egyptian Mathematical Society 2016; 24: 204-209.

[6] Zayed EME, Nowehy AGA. The Modified Simple Equation Method, the Exp-Function Method, and the Method of Soliton Ansatz for Solving the LongShort Wave Resonance Equations, Zeitschrift für Naturforschung A 2015; 71: 91-180.

[7] Mirzazadeh M. Modified Simple Equation Method and its Applications to Nonlinear Partial Differential Equations, Information Sciences Letters 2014; 1: 1-9.

[8] Amna Irshada A, Mohyud-Din TS, Ahmed N, Khanc U. A New Modification in Simple Equation Method and its applications on nonlinear equations of physical nature, Information Sciences Letters 2014; 1: 1-9.

[9] Younis M, Zafar A. The modified simple equation method for solving nonlinear Phi-Four equation , International Journal of Innovation and Applied Studies 2013; 4: 661-664.

[10] Lu D, Seadawy A, Arshada M. Applications of extended simple equation method on unstable nonlinear Schrdinger equations, Optik-International Journal for Light and Electron Optics 2017; 140: 136-144.

[11] Akcagi S, Aydemir T. Comparison between the (G/G) - expansion method and the modified extended tanh method, Open Physics 2016; 14: 88-94.

[12] Zhang W. A Generalized Tanh-Function Type Method and the (G/G) -Expansion Method for Solving Nonlinear Partial Differential Equations, Applied Mathematics 2013; 4: 1-16.

[13] Wang M, Xiangzheng L, Jinliang Z. The (G/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Physics Letters A 2008; 417-423.

[14] Manafian J. Optical soliton solutions for Schrdinger type nonlinear evolution equations by the $tan(\Phi(\xi)/2)$-expansion method, Optik 2016; 127: 4222-4245.

[15] Manafian J. Expanding the Tanh-Function method for solving nonlinear equations , Applied Mathematics, 2011; 2: 1096-1104.

[16] Panahipour H. Application of Extended Tanh Method to Generalized Burgers-type Equations , Communications in Numerical Analysis, 2012; 2012: 1-14.

[17] Khan K, Akbar MA. Application of Exp$(-phi(xi))$-expansion Method to Find the Exact Solutions of Modified Benjamin-Bona-Mahony Equation, World Applied Sciences Journal, 2013; 10: 1373-1377.

[18] Bekir A, Aksoy E. Application of the Subequation Method to Some Differential Equations of Time-Fractional Order, World Applied Sciences Journal, 2015; 10: 1-5.

[19] Ege SM, Misirli E. Traveling Wave Solutions of Some Fractional Differential Equations, Romanian Journal of Mathematics and Computer Science, 2016; 6: 106-115.

[20] Mahmud F, Samsuzzoha M, Akbar A. The generalized Kudryashov method to obtain exact traveling wave solutions of the PHI-four equation and the Fisher equation, Results in Physics, 2017; 7: 4296-4302.

[21] Kaplan M, Bekir A, Akbulut A. A generalized Kudryashov method to some nonlinear evolution equations in mathematical physics, Nonlinear Dynamics, 2016; 85: 2843-2850.

[22] Ege SM. Extended traveling wave solutions for some nonlinear equations, Advances in Mathematics: Scientific Journal - Research Publication, 2016; 5: 179-189.

[23] Guner O, Bekir A. A novel method for nonlinear fractional differential equations using symbolic computation, Waves in Random and Complex Media, 2017; 27: 163170.

[24] Arbabi S, Najafi M , Analytic Solution of the $(1+1)$-Dimensional Ito equation using the sine-cosine Method, International Journal of Modern Mathematical Sciences, 2012; 4: 89-95.

[25] Bhrawy AH, Alhuthali MS, Abdelkawy MA. New Solutions for (1+1)-dimensional and (2+1)-dimensional Ito Equations, Mathematical Problems in Engineering, 2012; 2012: 1-24.

[26] Gepreel KA, Nofal TA, Alasmari AA. Exact solutions for nonlinear integro-partial differential equations using the generalized Kudryashov method, Journal of the Egyptian Mathematical Society, 2017; 2017: 1-7.

Department of Mathematics
Ege University
Bornova, Izmir 35100 Turkey
E-mail address: serife.muge.ege@ege.edu.tr